The Review of Finance and Banking
Volume 04, Issue 1, Year 2012, Pages 055–073
S print ISSN 2067-2713, online ISSN 2067-3825

EVALUATING AN INVESTMENT PROJECT IN AN INCOMPLETE MARKET

GEORGE YUNGCHIH WANG

Abstract. Many studies on real options base their arguments upon the assumption that security market is complete to apply the risk-neutral valuation technique. However, when the market is incomplete, in which investment risk is not spanned by existing assets, the investor's preferences are not risk-neutral, or management is refrained from certain trading strategies, there may exist no unique martingale price of uncertain income streams. In this paper, a dynamic-programming framework for maximizing expected utility of an investor in discrete time is presented to evaluate an investment opportunity in an incomplete market. It is suggested that certainty equivalent (CE) could be applied to value such an investment opportunity. We show that two approaches to certainty equivalent, i.e., the buying price and the selling price approaches, are exactly equal in exponential utility, implying that CE is a fair value for both the buyer and the seller in an incomplete market, subject to the degree of risk aversion. Therefore, the proposed approach, compared to other alternative approaches, is relatively intuitive and easy to apply. With the classic investment problem, it is shown that the option embedded in a project is crucial in decision-making.

1. Introduction

The valuation of an irreversible investment project is a crucial topic in the field of finance and project management. Over the last three decades, real options theory has become the dominant focus of capital investment theory. Many real options models for evaluating capital investments explicitly and implicitly assume that the security market is complete to hedge away investment uncertainties in order to apply the risk-neutral valuation technique. This is equivalently to imply the assumption of the no-arbitrage principle, under which all securities have the same expected rate of return, namely, the riskless rate.

A complete market, by definition, describes the situation in that the payoffs of the focus asset can be perfectly replicated or 'spanned' by a single marketed asset (also called a twin asset) or an equivalent portfolio of securities. Under the complete market assumption, investors are able to hedge away the price risks of the focus asset in all states by dynamically trading marketed securities, and thus adopt the market's risk-neutral probabilities as their own to identify a unique no-arbitrage price or an equivalent martingale measure (see Harrison and Kreps, 1979), which is also known as risk-neutral valuation. The replicating portfolio approach originates from the seminal studies by Black and Scholes (1973) and Merton (1973) in valuing European options in continuous time and further advanced by Cox, Ross and Rubinstein (1979) and Rendleman and Bartter (1979) who value financial options with a binomial lattice in discrete time.

Market completeness is not readily obtained when the market has no sufficient securities to hedge away price risk. The risk-neutral probability measure in an incomplete market is thus

Received by the editors July 4, 2011. Accepted by the editors January 23, 2012.
Keywords: real options, incomplete market, income streams, certainty equivalent, dynamic programming.
JEL Classification: G11, G13, G19.

George Yungchih Wang , Ph.D., is assistant professor at the National Kaohsiung University of Applied Sciences, Taiwan. E-mail: gwang@cc.kuas.edu.tw.

This paper is in final form and no version of it will be submitted for publication elsewhere.

not unique to such an extent that the price of a contingent claim falls between the price bounds of the infimum and the supremum, which are determined from a collection of all available equivalent martingale measures.[1] Since risk-neutral valuation techniques cannot determine a unique measure for pricing a contingent claim in the relaxation of market completeness, recent research is thus directed at incorporating individual's risk preference and objective probability measures into the valuation framework of utility maximization.[2] Therefore, the valuation of income streams in an incomplete market may rely on the calculations of a certainty equivalent (CE) value either from the buying price approach or the selling price approach.[3]

This paper, drawn from the research on asset pricing in utility maximization frameworks, aims to develop a dynamic programming model for evaluating income streams generated from an investment project in an incomplete market in a discrete-time, discrete-space fashion. This paper postulates that the recursive structures of utility maximization over terminal wealth could be explicitly solved in order to evaluate income streams generated from an investment project in an incomplete market. Since the risk of income streams in an incomplete market could not be completely hedged away with existing traded securities, the concept of certainty equivalent, also known as the delta property, is introduced to evaluate an investment opportunity. The delta property states that the investor is indifferent between receiving uncertain cash flows and a certain amount.[4] By applying the delta property to utility maximization, we show that the certainty equivalent derived from the buying price approach is equal to that derived from the selling price approach when the utility function has a delta property. This equality relationship indicates that the certainty equivalent is a fair price for income streams should both of the buyer and the seller have the same risk attitude. Therefore, the approach proposed in this paper, compared to the other models, is relatively intuitive and easy to apply. In addition, by using this methodology, management would avoid to make an arbitrage estimate in risk-adjusted discount rate and thus reduce the chance of making mistaken decisions.

The rest of the paper is organized as follows: Section 2 reviews the literature on asset pricing in an incomplete market. Section 3 describes the dynamic programming framework, including the model specifications and the derivation of recursive structures. Numerical techniques are also developed to tackle the computation problem in the utility function. In Section 4, we extend the original problem to maximizing expected utility over consumptions over time with/out terminal wealth. In addition, we also propose a method to measure market completeness. Section 5 presents a classic investment problem in the context of real options. Since the same investment problem has been illustrated in many other papers, it is thus convenient to compare the different approaches to valuation. We then solve the classic investment problem with the proposed model and numerical techniques in comparison to traditional discounted cash flow (DCF) approach. Section 6 provides concluding remarks.

2. **Literature Review**

Since the seminal work in Black and Scholes (1973), most options models along the line of research implicitly or explicitly assume the existence of a complete market which allows for a perfect hedging to eliminate all the states of uncertainty of contingent claims. Harrison and Kreps (1979) consider this fundamental issue of arbitrage argument and find that in a complete market one may construct a risk-neutral probability measure from a linear state-price function for deriving a unique discounted asset price or a martingale. The risk-neutral probability measure, for this reason, is called the equivalent martingale measure. They also

[1] El-Karoui and Quenez (1995).

[2] See Smith and Nau (1995), Pliska (1997), and Smith (1998).

[3] The buying price is defined as the lump-sum present payment, made by the buyer, which makes expected utility with the project equal to expected utility without the project. The selling price is referred to as the added-in certainty equivalent, received by the seller, which makes expected utility without the project equal to expected utility with the project. See Smith and Nau (1995).

[4] See also the discussions in Howard (1971) and Luenberger (1998, Ch. 16).

make it clear that if the market is complete, the number of traded assets must be greater than or at least equal to the number of the states of the world. In an infinite-state world with no arbitrage opportunities allowed, Kreps (1981) shows how the martingale property of a price system can be determined in individual's expected utility representation. Duffie and Huang (1985) demonstrate that the martingale property may be effectively reached with dynamic spanning in infinite states and in continuous time with only a few long-lived securities. Sharing the same spirit from earlier studies, Nau and McCardle (1991) postulate that the martingale price in conventional option pricing models may be still valid as long as the no-arbitrage principle holds. Under the no-arbitrage condition, the technique of expected utility maximization may be introduced into the pricing. It is shown that the result in the no-arbitrage principle is effectively equivalent to that under the complete-market assumption except less information required.

Empirical evidence suggests that financial markets in reality are incomplete and thus impeding no-arbitrage pricing for a number of reasons. The first reason of market incompleteness is mainly caused by trading frictions such as transaction costs. (Aiyagari and Gertler, 1991; Aiyagari, 1993) Furthermore, model risk involved in risk management typically indicates a failure of market completeness. (Figlewski, 1998) Figlewski discusses several sources of model risk facing derivative traders, the misspecification of trading models, the difficulty of estimating input parameters, and the infeasibility of practical delta hedging. Green and Figlewski (2001) demonstrate that even though an institution uses an optimal delta hedging ratio as well as a correct volatility forecast in an attempt to hedge away the risk of writing an option, the firm still exposes a considerable risk to the financial market because of model risk, thus providing strong evidence for market incompleteness. Recently, Hugonnier and Morellec (2007) state an incomplete market, in addition to the insufficiency of existing assets, also involves management restrictions in that corporate executives are refrained from certain trading strategies. It is demonstrated that risk aversion may induce management to expedite investment, leading to an erosion of option value.

The main impediment to investment valuation in an incomplete market is that not all contingent claims are attainable with the replication of existing marketed securities, and, even though they are, there may not exist a unique equivalent martingale measure for the contingent claim in question. El-Karoui and Quenez (1995) find that the collection of all available equivalent martingale measures in the state space would form an interval of price bounds between the infimum and the supremum, both of which are also defined as the bid-ask spread to include all the possible prices in an incomplete market. The infimum, F_t^-, and the supremum, F_t^+, are expressed as follows, respectively:

$$F_t^- = \inf \left[E_t^Q \left(F_T \right) \right] \tag{1}$$

$$F_t^+ = \sup \left[E_t^Q \left(F_T \right) \right] \tag{2}$$

where $E_t^Q \left(F_T \right)$ denotes the expected value of the contingent claim at time t expired at time T. with Q representing all the available equivalent martingale measures in an incomplete market. (See also Pliska, 1997)

Some early financial economists, in fact, may have noticed the pricing problem in an incomplete market, but mostly they adjust the drift rate of underlying assets in order to continue assuming complete market. For example, Brennan and Schwartz (1982) and Cox, Ingersoll, and Ross (1985) adjust the drift rate of each stochastic process by an amount of a process-specific risk premium where the risk premium is derived from the equilibrium model of financial markets. By this justification, the security market is seen as if it is complete.

To tackle the difficulty of pricing contingent claims in an incomplete market, El-Karoui and Quenez (1995) describe a superhedging technique that the seller may purchase a replicating portfolio at the supremum price to hedge the contingent claim and lock in a profit. The problem of such a technique is that the superhedging strategies may require a large amount of initial capital, which becomes a dilemma for the investors who are unwilling to put in the initial capital. Follmer and Leukert (1999) propose an alternative technique called quantile hedging by constructing a hedging strategy which maximizes the probability of a successful hedge under the objective probability measure, given a constraint on the required cost. Another alternative technique is called good-deal pricing, proposed by Cochrane and Saa-Requejo (2000) and Cerny and Hodges (2000). The basic intuition of good-deal pricing is that investors not only exploit all the arbitrage opportunities but also search for hedging the opportunities of good deals, expressed by assets with a high Sharpe ratio. Implementing the good-deal hedging would greatly shrink the price bounds in an incomplete market.

Another approach to asset pricing in an incomplete market is proposed by Smith and Nau (1995), who integrate the utility maximization technique and the decision tree analysis to evaluate an investment opportunity. They assume that project uncertainties can be classified into market uncertainty (or price uncertainty) and private uncertainty (or technological uncertainty); the former can be fully resolved by the market with a unique risk-neutral probability measure and the latter is resolved with a risk-adjusted probability measure. The utility maximization technique is then introduced to the framework to explicitly model investor's risk preference. This particular assumption of uncertainty makes the integrated framework become a special case in the incomplete-market pricing. Furthermore, similar to Nau and McCardle (1991), Smith and Nau assume that investor's objective is to maximize expected utility over terminal wealth only. However, if investor's fundamental preference is for consumption, we should maximize expected utility over consumption rather than terminal wealth. Smith (1998) takes consumption streams into the optimization framework, yet he still makes the same specific assumption regarding project uncertainty as in Smith and Nau (1995). Staum (2004) develops a method of marginal indifference pricing which provides a unique price based on expected utility in the absence of exact hedging replication, but the methodology may be subject to model misspecification. Recently, Pyo (2008) explores the real option problem in an incomplete market within a hyperbolic absolute risk aversion (HARA) utility function to derive narrower price bounds.

While the above line of research attempts to form a hedging strategy to price a contingent claim in an incomplete market, other researchers in real options focus on searching for highly correlated marketed assets to complete the market. For example, Teisberg (1995) suggests to identify a twin asset by choosing the publicly traded firms whose projects are similar to those of interest. Alternatively, a portfolio of traded assets or cash flows of a completed project can be treated as a twin asset. A similar idea to the twin asset, proposed by Sick (1995), is termed the Hotelling valuation principle which introduces a highly correlated commodity. In principle, the Hotelling valuation implies that the unit value of exhaustible natural resources can be written as a function of its current price, net of extraction cost; other variables like interest rate have no additional explanatory power. With the Hotelling valuation principle, we can relate the value of an underlying asset to the spot (or future) price of a certain correlated commodity and use it as the underlying asset. However, the problem of these methods arises when practitioners in reality find it so difficult to discover the highly correlated assets with the same risk profile as the investment project in question.

Copeland, Koller, and Murrin (2000) and Copeland and Antikarov (2001)[5] suggest the Marketed Asset Disclaimer (MAD) assumption, which states that the present value of the project itself, without flexibility, is the best unbiased estimator of the market value of the project if it should be traded on the market. Since the project without flexibility is assumed to be highly correlated with the project with flexibility, no other securities on the market can make a better twin asset than it is. The intuition behind the MAD assumption is easily understood, but what makes it difficult to apply is that securities markets are assumed to be complete such that the payoffs of the project with no flexibility can also be replicated by the stocks. If the twin assets do not exist in reality, theoretically fictitious securities could also be introduced to complete

[5]See Copeland et al. (2000), pp. 406-407; Copeland and Antikarov (2001), pp. 94-95.

the model and then impose the restrictions to prohibit investors from holding any positions in the fictitious securities. (Pliska, 1997) The major drawback in conducting the technique of fictitious securities is that this technique becomes computationally expensive as the number of states of the world is very large and there are only a few marketed securities available.

The idea of evaluating uncertain income streams in an incomplete market by a CE value has long been adopted theoretically and empirically in finance literature. For instance, Silvers (1973) examined the placecountry-regionU.S. investment-grade bond prices for the period of 1952-1964 within the utility-maximization framework and found empirical support for certainty equivalent. Jacque (1980) suggested an expected utility framework for foreign exchange decisions. With the data of US foreign exchange rates, she derived an explicit relationship between the level of risk aversion and the CE value. In addition, O'Brien (1992) used the idea of CE to test the empirical validity of the preference-inducing technique and found that most transactions were individually rational with respect to their predicted CEs. More recently, Hagstromer, Anderson, Binner, Elger, and Nilsson (2008) used a portfolio choice setting of three country-regionplaceUK indices to identify CEs in several utility functions. Furthermore, Fugazza, Guidolin, and Nicodano (2009) found that diversification into real estate investments (REITs) increases both the Sharpe ratio and the CE of wealth for all investment horizons.

In addition to evaluating investor's wealth in stock markets, certainty equivalence was also found empirical support for stochastic commodity prices. For example, Valenzuela and Mazumdar (2003) applied the CE idea to evaluate the option of trading electricity at market prices. Serra, Goodwin, and Featherstone (2011) applied an expected utility model to empirically evaluate farmer's CE and the risk-related effect s of farm policy changes at the intensive margin of production.

Building on the approach of dynamic programming, this paper further extends the idea of utility maximization in Wang (2010) to the objective of maximizing expected utility over consumption with/out terminal wealth in an incomplete market. Wang (2010) dealt with the problem of investment evaluation under the assumption that the objective of a decision-maker is to maximize expected utility over terminal wealth while this paper relaxed the assumption to a more realistic case in which the decision-maker aims to maximizing expected utility over consumption with/out terminal wealth. Furthermore, we develop a technique of applying the coefficient of correlation to measure market incompleteness in this paper.

3. Methodology

3.1. **Model Specifications.** Suppose a finite sample space, Ω , exists with $k(k < \infty)$, elements, $\Omega = \{\omega_1, \omega_2, ..., \omega_k\}$, where ω , denotes the state of the world. All the investments and consumptions take place at time $t = \{0, 1, ..., T\}$. There is a probability measure P on Ω, with $P(\omega) \geq 0$ for all $\omega \epsilon \Omega$ and $\sum_{i=1}^{k} P(\omega_i) = 1$.

Securities markets are assumed to be frictionless to such an extent that the investor can trade any amount of shares of a security at a market price without incurring any transaction cost. S denotes a security price process, $S = \{S_t^n : t = 0, 1, ... T; n = 0, 1, ..., N\}$, where S is a security matrix and S_t^n is a scalar representing the price of security n at time t. Among all the securities, S^0 denotes a risk-free security vector while the others denote risky security vectors. Let θ be a vector of trading strategies, $\theta = \{\theta_t^n : t = 0, 1, ...T; n = 0, 1, ...N\}$, in the investor's portfolio where the scalar θ_t^n represents the units of security n held between time t and $t + 1$ and the scalar θ_t^0 is the dollar amount invested S_t^0 in from time t to $t + 1$ at a risk-free rate, r. In addition, x denotes a security return process, $x = \{x_t^n : t = 0, 1, ..., T; n = 0, 1, ..., N\}$, where x_t^n is the return of security n from time $t - 1$ to time t.

Investor's wealth is denoted by a value process, $V = \{V_t : t = 0, 1, ..., T\}$, which represents the total value of the portfolio at time t. The consumption plans, $C = \{C_t : t = 0, 1, ..., T\}$, are a non-negative stochastic process with C_t representing the amount of funds consumed by the investor at time t. Note that the consumption process is admissible when $C_t \leq V_t$. Also, at the end of the intended time horizon, all the terminal wealth is consumed so that $C_T = V_T$. Suppose there is a process of cash streams, $H = \{H_t : t = 0, 1, ...T\}$ where $H_t(\omega)$ denotes income streams at time t and state ω. Therefore, the wealth, $V_t(\omega)(\forall\omega\epsilon\Omega, t \geq 1)$, can be expressed as follows:

$$V_t(\omega) = (V_{t-1} - C_{t-1})(1+r) + \theta_{t-1}S_t(\omega)[x_t(\omega) - r] + H_t(\omega), \ \forall\omega \in \Omega^6 \quad (3)$$

where r is risk-free rate and $x_t(\omega) = \frac{S_t(\omega) - S_{t-1}}{S_{t-1}}$.
If there is no consumption over time, Equation (3) can be modified as follows:

$$V_t(\omega) = V_{t-1}(1+r) + \theta_{t-1}S_t(\omega)[x_t(\omega) - r] + H_t(\omega), \forall\omega\epsilon\Omega^7 \quad (4)$$

When securities markets are pronounced complete, it is equivalent to assuming the existence of the no-arbitrage condition. This means there is no arbitrage opportunity so that the investor cannot profit from trading securities without taking some risk or spending some capital. An alternative setup of this no-arbitrage condition is to regard the existence of risk-neutral probabilities. The law of one price permits us to derive exactly one set of risk neutral probability measures[8]. Let Q represent a risk-neutral probability measure on Ω , with $Q(\omega)$ for all $\omega\epsilon\Omega$. Securities markets are arbitrage-free if and only if a strictly positive risk-neutral probability measure exists in the following way:

$$S_0 = \frac{\sum_{\omega\in\Omega} Q(\omega) S_t(\omega)}{(1+r)^t} = \frac{E_Q S_t}{(1+r)^t}, \quad (5)$$

By contrast, there is no unique risk-neutral probability measure in an incomplete market so that Equation (5) can be modified as follows:

$$S_0 = \frac{\sum_{\omega\in\Omega} Q^m(\omega) S_t(\omega)}{(1+r)^t} = \frac{E_{Q^m} S_t}{(1+r)^t} \quad (6)$$

where m denotes the finite sets of risk-neutral probability measures.

For time period t, the market is incomplete if $N+1 < k$, meaning that existing marketed securities cannot replicate the payoffs of a contingent claim for all the states of the world. In a matrix form, we cannot solve the following system of linear equations for a unique solution of θ:

$$\begin{bmatrix} (1+r) & S^1(\omega_1) & \cdots & S^N(\omega_1) \\ (1+r) & S^1(\omega_2) & \cdots & S^N(\omega_2) \\ \vdots & \vdots & \vdots & \vdots \\ (1+r) & S^1(\omega_k) & \cdots & S^N(\omega_k) \end{bmatrix} \times \begin{bmatrix} \theta^0 \\ \theta^1 \\ \vdots \\ \theta^N \end{bmatrix} \neq \begin{bmatrix} H(\omega_1) \\ H(\omega_2) \\ \vdots \\ H(\omega_k) \end{bmatrix} \text{ if } N+1 < k \quad (7)$$

In the context of investment valuations, an investment opportunity in complete market is seen as a redundant asset so that its payoffs can be fairly replicated with existing marketed securities. However, in incomplete market, the payoffs of the investment opportunity cannot be completely hedged away with the marketed securities so that replicating residuals must exist for at least one state of uncertainty. Let ε denote a vector of replicating residuals in that the scalar $\varepsilon_t(\omega)$ denote the replicating residual at time $t$for all $\omega\epsilon\Omega$. Thus, the income streams generated from the investment opportunity can be expressed as follows:

$$H_t(\omega) = \theta^0_{t-1}(1+r) + \sum_{i=1}^{N} \theta^i_{t-1} S^i_t(\omega) + \varepsilon_t(\omega), \ \forall\omega \in \Omega \quad (8)$$

---

[6]If there are no cash streams, Equation (3) becomes the following form: $V_t(\omega) = (V_{t-1} - C_{t-1})(1+r) + \theta_{t-1}S_t(\omega)[x_t(\omega) - r]$.

[7]If there are no cash streams, Equation (4) becomes the following form: $V_t(\omega) = V_{t-1}(1+r) + \theta_{t-1}S_t(\omega)[x_t(\omega) - r]$.

[8]Pliska (1997).

n complete markets, only one set of trading strategies can fully hedge the income streams so that $\varepsilon_t(\omega) = 0, \forall \omega \epsilon \Omega$. By contrast, since project payoffs can not be fully hedged in complete markets, at least one non-zero $\varepsilon_t(\omega)$ exists, i.e., $\sum\limits_{\omega \epsilon \Omega} \varepsilon_t(\omega) \neq 0$.

3.2. **Dynamic Programming.** For our recursive optimization problems, exponential utility is introduced into the objective function. The exponential utility function is chosen for at least three special reasons. Firstly, it demonstrates the property of constant absolute risk aversion (CARA) which makes the certainty equivalent and the optimal investment decision independent of the level of initial wealth, and the exponential utility also allows us to readily observe how the certainty equivalent is influenced by the investor's risk preference, expressed by the coefficient of CARA. Secondly, in the exponential utility function, both the temporal value function, J_t , and the temporal consumption, C_t , can be explicitly expressed as a function of wealth, V_t as recursive structures while other forms of utility functions can not derive such convenient solutions. Thirdly, the delta property of the exponential utility makes the certainty equivalent of the buying price approach equal to that of the selling price approach. The use of exponential utility in literature is adequate, such as Davis (2001) in financial planning, Bliss and Panigirtzoglou (2004) in option pricing, and Henderson (2005) in portfolio choice. As argued by Merton (1992), optimal consumption in exponential utility is linear in wealth and thus consistent with a balanced growth path.

Suppose the investor's objective is to maximize his/her expected utility over terminal wealth (V_T). To simplify the analysis, we let α denote the position invested in the risky security, i.e., $\alpha = \theta S$, be the matrix of excess return of risky stocks, i.e., $X = x - r$, and A be the coefficient of the investor's risk attitude ($A > 0$) . The utility maximization problem can be formulated as follows:

$$\begin{aligned} &\max_{\alpha_0^*, \alpha_1^*, \cdots, \alpha_{T-1}^*} E_0 \left[-\exp\left(-AV_T\right) \right] \\ &\text{s.t. } V_t = (1+r)\, V_{t-1} + \alpha_{t-1}^* X_t + H_t,\ t = 1, \cdots, T \end{aligned} \tag{9}$$

Let the superscript * denote the parameters affected by the income streams. With the rule of iterative expectations, the optimization problem can be expressed as the following dynamic programming equation:

$$J_0^* = \max_{\alpha_0^*, \alpha_1^*, \cdots, \alpha_{T-2}^*} E_0[J_{T-1}^*] \tag{10}$$

where J denotes the maximized utility function and $J_{T-1}^* = \max\limits_{\alpha_{T-1}^*} E_{T-1}\left[-\exp\left(-AV_T\right)\right].$

Substituting the budget constraint into Equation (10) and taking the first order condition with respect to α_{T-1}^* equal to zero, we know that α_{T-1}^* can be solved from:

$$E_{T-1}\left\{X_T \exp\left[-A\left(\alpha_{T-1}^* X_T + H_T\right)\right]\right\} = 0 \tag{11}$$

or alternatively

$$\sum_{\omega \in \Omega} P(\omega)\, X_T(\omega) \exp\left[-A\left(\alpha_{T-1}^* X_T(\omega) + H_T(\omega)\right)\right] = 0 \tag{12}$$

By recursive optimizations for multi-periods, it is convenient to generalize that for any time period t, α_t^* can be solved from:

$$E_t\left\{\beta_{t+1}^* X_{t+1} \exp\left[-A\lambda_{t+1}\left(\alpha_t^* X_{t+1} + H_{t+1}\right)\right]\right\} = 0 \tag{13}$$

where $\beta_t^* = E_t(K_{t+1}^* \beta_{t+1}^*), \beta_T^* = 1$,
$K_t^*(\omega) = \exp\{-A\lambda_t[\alpha_{t-1}^* X_t(\omega) + H_t(\omega)]\}, \forall \omega \epsilon \Omega$, and

$$\lambda_t = (1+r)^{T-t}.$$

The temporal value function for any time period t can be expressed as

$$J_t^* = \beta_t^*\left[-\exp\left(-A\lambda_t V_t\right)\right] \tag{14}$$

Next, the same utility maximization problem needs to be solved without income streams in the budget constraints. The problem in Equation (9) thus is restated as follows:

$$\begin{aligned} &\max_{\alpha_0,\alpha_1,\cdots,\alpha_{T-1}} E_0\left[-\exp\left(-AV_T\right)\right] \\ &\text{s.t. } V_t = (1+r)\,V_{t-1} + \alpha_{t-1}X_t,\ t = 1,\cdots,T \end{aligned} \tag{15}$$

The recursive structures in this problem are exactly the same as Equation (13) and Equation (14) with the asterisk, * , removed. Since there are no income streams involved in the problem, α_t must be solved from:

$$E_t\left[\beta_{t+1}X_{t+1}\exp\left(-A\lambda_{t+1}\alpha_t X_{t+1}\right)\right] = 0 \tag{16}$$

Note that the main difference between α_t^* and α_t is that the former is the optimal trading strategies under the consideration of income streams while the latter represents the autonomous trading strategies without income streams. The temporal value function associated with Equation (15) can be solved as given below:

$$J_t = \beta_t\left[-\exp\left(-A\lambda_t V_t\right)\right] \tag{17}$$

Note that Equations (14) and (17) have the same functional form. Next, we introduce two approaches to solve for certainty equivalent.

3.3. **Solving for Certainty Equivalent.** The basic idea of a certainty equivalent is to calculate the dollar amount that equates both maximized utility functions, J and $J^\bullet$. This idea can be conceptualized in Figure 1.

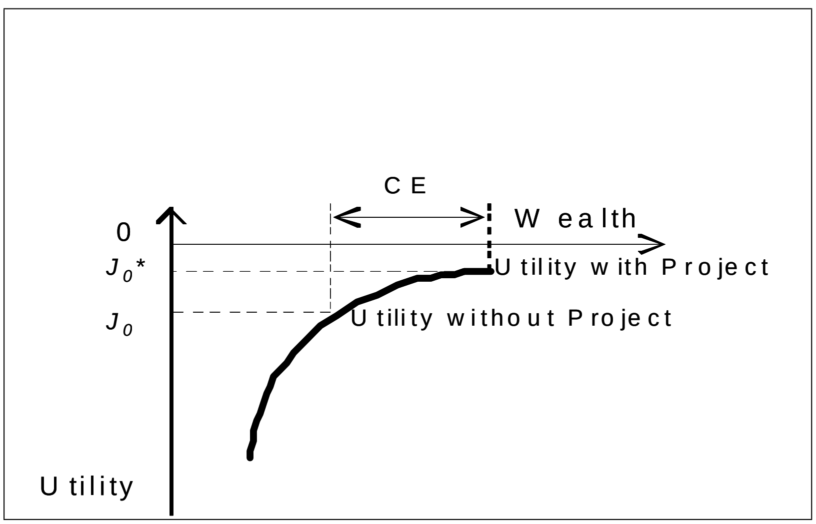

Figure 1. The Conceptual Framework for Deriving Certainty Equivalent (CE)

As mentioned earlier, there are two approaches to the derivation of certainty equivalent, the buying price approach and the selling price approach. The buying price is defined as the lump-sum present payment, made by the buyer, which makes the maximum expected utility with the project, J_0^* , equal to the maximum expected utility without the project, J_0. Let CE_t^B denote the certainty equivalent of income streams, based on the buying price approach at time t. Thus, the buying price approach describes the following relationship at time t:

$$\beta_t^*\left\{-\exp\left[-A\lambda_t\left(V_t - CE_t^B\right)\right]\right\} = \beta_t\left\{-\exp\left[-A\left(\lambda_t V_t\right)\right]\right\} \tag{18}$$

When solving Equation (18) for CE_t^B, we have the certainty equivalent of income streams as shown below:

$$CE_t^B = \frac{1}{A\lambda_t}\left[\ln\left(\frac{\beta_t}{\beta_t^*}\right)\right] \tag{19}$$

The selling price is defined as the added-in certainty equivalent, received by the seller, which makes the maximum expected utility without the project equal to the maximum expected utility with the project in order to compensate for the loss of the project. Let CE_t^S denote the certainty equivalent of income streams, based on the selling price approach at time t. By equating the expected utilities both with and without the project, we have the following relationship:

$$\beta_t^* \{-\exp[-A(\lambda_t^* V_t)]\} = \beta_t \left\{-\exp[-A\lambda_t(V_t + CE_t^S)]\right\} \tag{20}$$

Solving the above equation for CE_t^S, we have

$$CE_t^S = \frac{1}{A\lambda_t}\left[\ln(\frac{\beta_t}{\beta_t^*})\right] \tag{21}$$

One could easily find that Equations (19) and (21) are exactly equivalent. The implication of this equality makes sure that CE^B or CE^S is a fair price for income streams of an investment opportunity to both the buyer and the seller when they have the same risk preference. In addition, compared to earlier related studies, the result of the equality of the buying price and the selling price provides an improvement because the solution of certainty equivalent is explicitly given as well as shown to be negatively dependent on the investor's risk attitude. The negativity is interpreted in the sense that when A becomes larger, i.e., the investor is more risk-averse, the certainty equivalent is smaller. The main result of dynamic programming is summarized in Table 1.

Table 1. Maximizing Expected Utility over Terminal Wealth

(A)

Optimization Objective	Terminal Wealth (Without Income Streams in the Constraint)
Problem	$\mathrm{J}_0=\max_{\alpha_0,\alpha_1,\cdots,\alpha_{T-1}}\mathrm{E}_0\left[-\exp\left(-AV_T\right)\right]$
Constraint	$\mathrm{V}_t\left(\omega\right)=(1+r)\mathrm{V}_{t-1}+\alpha_{t-1}\mathrm{X}_t\left(\omega\right),\ \forall\omega\in\Omega$
Solution for the Last Period	α_t is solved from $\mathrm{E}_t\left[\beta_{t+1}X_{t+1}\exp\left(-A\lambda_{t+1}\alpha_t X_{t+1}\right)\right]=0,\ \lambda_T=1,\ \beta_T=1$
Recursive Structures	$\lambda_t=(1+r)^{T-t}$, $\mathrm{K}_t\left(\omega\right)=\exp\left[-A\lambda_t\alpha_{t-1}X_t\left(\omega\right)\right]$, $\beta_t=\mathrm{E}_t\left(K_{t+1}\beta_{t+1}\right)$, $\mathrm{J}_t=\beta_t\left[-\exp\left(-A\lambda_t V_t\right)\right]$
Maximized Expected Utility	$\mathrm{J}_0=\beta_0\left[-\exp\left(-A\lambda_0 V_0\right)\right]$
Certainty Equivalent	$\mathrm{CE}_t=\frac{1}{A\lambda_t}\left[\ln(\frac{\beta_t}{\beta_t^*})\right]$

(B)

Optimization Objective	Terminal Wealth (With Income Streams in the Constraint)
Problem	$\mathrm{J}_0=\max_{\alpha_0^*,\alpha_1^*,\cdots,\alpha_{T-1}^*}\mathrm{E}_0\left[-\exp\left(-AV_T\right)\right]$
Constraint	$\mathrm{V}_t\left(\omega\right)=\mathrm{V}_{t-1}\left(1+r\right)+\alpha_{t-1}^*\mathrm{X}_t\left(\omega\right)+\mathrm{H}_t\left(\omega\right)$, $\forall\omega\in\Omega$,t=0,$\cdots$,T
Solution for the Last Period	α_t^* is solved from $\mathrm{E}_t\left\{\beta_{t+1}^*X_{t+1}\exp\left[-A\lambda_{t+1}\left(\alpha_t^*X_{t+1}+H_{t+1}\right)\right]\right\}=0$, $\lambda_T=1$, $\beta_T^*=1$
Recursive Structures	$\mathrm{K}_t^*\left(\omega\right)=\exp\left\{-A\lambda_t\left[\alpha_{t-1}^*X_t\left(\omega\right)+H_t\left(\omega\right)\right]\right\}$, $\forall\omega\in\Omega$, $\lambda_t=(1+r)^{T-t}$, $\beta_t^*=\mathrm{E}_t\left(K_{t+1}^*\beta_{t+1}^*\right)$, $\mathrm{J}_t^*=\beta_t^*\left[-\exp\left(-A\lambda_t V_t\right)\right]$
Maximized Expected Utility	$\mathrm{J}_0^*=\beta_0^*\left[-\exp\left(-A\lambda_0 V_0\right)\right]$
Certainty Equivalent	$\mathrm{CE}_t=\frac{1}{A\lambda_t}\left[\ln(\frac{\beta_t}{\beta_t^*})\right]$

3.4. **Numerical Techniques.** One possible problem encountered in the preceding derivation is that the magnitude of income streams may be too large to compute in the exponential utility function. Under the circumstances, income streams could be resolved with existing traded securities into the form of Equation (8) and the hedged residuals could be handled in a simple fashion. The basic intuition underlying the numerical techniques is to truncate the additional states of the world to match existing traded securities for the purpose of solving the system of equations for one set of trading strategies. For the world with *N+1* securities and k risky states, $(N+1<k)$, we need to leave $k-(N+1)$ states out of the system and solve a system of N equations for the unique solution of trading strategies, $\theta=\{\theta^n : n=0,1,\cdots,N\}$:

$$\begin{bmatrix} (1+r) & S^1(\omega_1) & \cdots & S^N(\omega_1) \\ (1+r) & S^1(\omega_2) & \cdots & S^N(\omega_2) \\ \vdots & \vdots & \vdots & \vdots \\ (1+r) & S^1(\omega_{N+1}) & \cdots & S^N(\omega_{N+1}) \end{bmatrix} \times \begin{bmatrix} \theta^0 \\ \theta^1 \\ \vdots \\ \theta^N \end{bmatrix} = \begin{bmatrix} H(\omega_1) \\ H(\omega_2) \\ \vdots \\ H(\omega_{N+1}) \end{bmatrix} \tag{22}$$

Apparently, the residual terms, $\varepsilon(\omega_n), \forall n=0,1,\cdots,N$ will be equal to zero, but the truncated residuals, $\varepsilon(\omega_n), \forall n=N+1,\cdots,k$, will not be zero under the particular set of trading strategies. The magnitude of these non-zero residuals permits us to gain insight into the degree of market incompleteness.

In the world of one risk-free security and one risky security, income streams can be expressed as follows:

$$H_t(\omega)=\theta^0_{t-1}(1+r)+\theta^1_{t-1}S^1_t(\omega)+\varepsilon_t(\omega),\ \forall\omega\in\Omega \tag{23}$$

In addition, the replicating strategies are given as follows:

$$\theta^0_{t-1}=\frac{H_t(\omega_2)S_t(\omega_1)-H_t(\omega_1)S_t(\omega_2)}{(1+r)\left[S_t(\omega_1)-S_t(\omega_2)\right]} \tag{24}$$

$$\theta^1_{t-1}=\frac{H_t(\omega_1)-H_t(\omega_2)}{S_t(\omega_1)-S_t(\omega_2)} \tag{25}$$

In terms of risk-free and excess returns, $H_t(\omega)$ can be rearranged as follows:

$$H_t(\omega)=\tilde{H}_{t-1}(1+r)+\alpha^h_{t-1}X_t(\omega)+\varepsilon_t(\omega) \tag{26}$$

where $\tilde{H}_{t-1}=\theta^0_{t-1}+\theta^1_{t-1}S_{t-1}$ and $\alpha^h_{t-1}=\theta^1_{t-1}S_{t-1}$.

The sum of $\tilde{H}_{t-1}$ and α^h_{t-1} represents the hedged position with existing traded securities, where $\tilde{H}_{t-1}$ is the risk-free hedged component and α^h_{t-1} is the excess-return hedged component. In a world of one risk-free security and several risky securities, $\tilde{H}_{t-1}$ and α^h_{t-1} simply become a risk-free hedged vector and an excess-return hedged vector, respectively.

By substituting Equation (26) into the budget constraint in Equation (9), the wealth process can also be expressed by the sum of the risk-free term, the excess-return term, and the residual term as in Equation (26):

$$V_t(\omega)=(1+r)\left(V_{t-1}+\tilde{H}_{t-1}\right)+\tilde{\alpha}^*_{t-1}X_t(\omega)+\varepsilon_t(\omega) \tag{27}$$

where $\tilde{\alpha}^*_{t-1}$ denotes the new trading strategies and $\tilde{\alpha}^*_{t-1}=\alpha^*_{t-1}+\alpha^h_{t-1}$.

With Equation (27) substituted into the utility maximization problem as the new budget constraint, the optimization problem is easily solved for the certainty equivalent. For the problem of maximizing expected utility over terminal wealth, $\tilde{\alpha}^*_t$ can be solved from

$$E_t\left\{\beta_{t+1}X_{t+1}\exp\left[-A\lambda_{t+1}\left(\tilde{\alpha}^*_tX_{t+1}+\varepsilon_{t+1}\right)\right]\right\}=0 \tag{28}$$

where $\beta^*_t=E_t\left(K^*_{t+1}\beta^*_{t+1}\right)$, $\beta^*_T=1$, and

$$K^*_t(\omega)=\exp\left\{-A\lambda_t\left[\tilde{\alpha}^*_{t-1}X_t(\omega)+\varepsilon_t(\omega)\right]\right\},\forall\omega\in\Omega.$$

Finally, certainty equivalent at time t becomes

$$CE_t = \tilde{H}_t + \frac{X_{t+1}\left(\tilde{\alpha}_t^* - \alpha_t\right)}{(1+r)} + \frac{\varepsilon_{t+1}}{(1+r)} \tag{29}$$

The expression of Equation (29) has an important implication for pricing income streams. As we already know, the first term represents the portion of certainty equivalent of income streams that can be hedged away with existing traded securities, and the second term stands for the difference in the risky position due to holding the investment opportunity. The last term is the replicating residuals yet to be resolved in the optimization due to market incompleteness. In the case of complete markets, i.e., the replicating residuals equal to zero, the certainty equivalent converges to a unique hedged value, $\tilde{H}_t$, plus the changes in the risky position due to holding the investment opportunity, $\frac{X_{t+1}(\tilde{\alpha}_t^* - \alpha_t)}{(1+r)}$. Consequently, with all the uncertain states of the world resolved, the probability measure P becomes an equivalent martingale measure and the certainty equivalent is thus a martingale price. By contrast, as the market is incomplete, it is impossible to pinpoint an exact present value of income streams, instead the certainty equivalent of income streams is situated in an interval of values which solely depends on the degree of market incompleteness, i.e., the residuals. Finally, note that the coefficient of risk aversion is dropped out of Equation (29), the derivation of certainty equivalent is independent of the degree of risk aversion.

To sum up, the preceding numerical technique can remove the hedged position, $\tilde{H}_t$, from the optimization procedure and allow us to simply deal with the hedged residuals for easier computation. More importantly, the numerical technique provides a way to observe how project values converge to a martingale and diverge into an interval of price bounds dependent on the replicating residuals. The magnitude of the replicating residuals sheds light on the degree of market incompleteness.

3.5. **The Value of an Investment Project.** Now that the value of an investment opportunity could be identified from the certainty equivalent, it is thus clear that net present value of an "invest now" alternative could be expressed as follows:

$$NPV_{invest} = CE_t - I \tag{30}$$

In a similar fashion, it is convenient to calculate net present value of a "defer" alternative. Therefore, the value of a deferral option, $V_{deferral}$, is computed as follows:

$$V_{deferral} = NPV_{defer} - NPV_{invest} \tag{31}$$

4. Extensions

The derivations in the preceding section are based on the maximization of expected utility over terminal wealth. In fact, an individual investor may be interested in maximizing expected utility over consumptions over time with/out terminal wealth. In this section, we would like to deal with the problem and derive certainty equivalent with the same approach of dynamic programming. Since certainty equivalent may converge to a martingale value in a complete market, its value depends on the degree of market completeness. Here, we would like to propose a method to measure market incompleteness.

4.1. **Maximizing Expected Utility over Consumptions with/out Terminal Wealth.** When individual investor's objective is to maximize expected utility over consumptions over time, i.e. withdrawing money from wealth each period, the utility function should be modeled explicitly over consumption streams. With the exponential utility function, the optimal investment-consumption problem can be formulated as follows:

$$\max_{\substack{\alpha_0^*, \alpha_1^*, \cdots, \alpha_{T-1}^* \\ C_0, C_1, \cdots, C_{T-1}}} E_0 \left\{ \sum_{t=0}^{T} B_t \left[-\exp\left(-AC_t\right) \right] \right\} \tag{32}$$

$$\text{s.t. } V_t(\omega) = (V_{t-1} - C_{t-1})(1+r) + \alpha^*_{T-1} X_t(\omega) + H_t(\omega), \forall \omega \in \Omega,$$
$$C_t \le V_t, t < T$$
$$C_T = V_T$$

where B_t denotes the utility weight at time t to capture investor's time preference for consumptions.

To simplify the problem, Equation (32) can be expressed as the dynamic programming equation, J_0, as follows:

$$\max_{\substack{\alpha^*_0, \alpha^*_1, \cdots, \alpha^*_{T-2} \\ C_0, C_1, \cdots, C_{T-1}}} E_0 \left\{ \sum_{t=0}^{T-2} B_t \left[-\exp\left(-AC_t\right)\right] + J_{T-1} \right\} \tag{33}$$

where $J_{T-1} = \max\limits_{\alpha^*_{T-1}, C_{T-1}} E_{T-1} \left\{ B_{T-1} \left[-\exp\left(-AC_{T-1}\right)\right] + B_T \left[-\exp\left(-AC_T\right)\right] \right\}$.

Recursive optimization of dynamic programming starts from working on J^*_{T-1}, taking the first order conditions with respect to α^*_{T-1} and C_{T-1}, and then moving backwardly until J^*_0. For period *T-1*, the first order condition ensures that α^*_{T-1} can be solved from

$$E_{T-1} \left\{ X_T \exp\left[-A \left(\alpha^*_{T-1} X_T + H_T \right) \right] \right\} = 0 \tag{34}$$

The optimal consumption at time $T - 1$ is solved in terms of a linear function of V_{T-1} as follows:

$$C_{T-1} = \gamma^*_{T-1} + \lambda_{T-1} V_{T-1} \tag{35}$$

where $\gamma^*_{T-1} = -\frac{\ln\left(\frac{B_T E_{T-1}(K^*_T)(1+r)}{B_{T-1}} \right)}{A(2+r)}$,

$$K^*_T(\omega) = \exp\left\{ -A \left[\alpha_{T-1} X_T(\omega) + H_T(\omega) \right] \right\}, \ \forall \omega \in \Omega,$$
$$\lambda_{T-1} = \frac{1+r}{2+r}.$$

Finally, the maximized utility function, J_{T-1}, can be derived as follows:

$$J^*_{T-1} = \beta^*_{T-1} \left[-\exp\left(-A\lambda_{T-1} V_{T-1} \right) \right] \tag{36}$$

where $\beta^*_{T-1} = B_{T-1} \exp\left(-A\gamma_{T-1} \right) + B_T E_{T-1}\left(K^*_T \right) \exp\left[A\gamma_{T-1}(1+r) \right]$.

Note that we still use the superscript * to note the parameters affected by income streams. For other periods, α^*_t can be found from

$$E_t \left\{ \beta^*_{t+1} X_{t+1} \exp\left[-A\lambda_{t+1}\left(\alpha^*_t X_{t+1} + H_{t+1} \right) \right] \right\} = 0 \tag{37}$$

where $\beta^*_t = B_t \exp\left(-A\gamma_t \right) + \beta^*_{t+1} E_t\left(K^*_{t+1} \right) \exp\left[A\lambda_{t+1}\gamma_t (1+r) \right]$.

The recursive structures of the solution for the other periods can be generalized as follows:

$$C_t = \gamma^*_t + \lambda_t V_t \tag{38}$$

where $\gamma^*_t = -\frac{\ln\left[\frac{\beta^*_{t+1} E_t\left(K^*_{t+1} \right) \lambda_{t+1}(1+r)}{B_t} \right]}{A[1+\lambda_{t+1}(1+r)]}$,

$K^*_t(\omega) = \exp\left\{ -A\lambda_t \left[\alpha^*_{t-1} X_t(\omega) + H_t(\omega) \right] \right\}$, and

$$\lambda_t = \frac{\lambda_{t+1}(1+r)}{1+\lambda_{t+1}(1+r)}, \lambda_T = 1.$$

It is worth noting that the solution of the maximized expected utility, J^*, is exactly the same as that of the problem of maximizing expected utility over terminal wealth.

In the case that no income streams are involved, the previous recursive structure can still work by assuming $H_t = 0, \ \forall t = 0, \cdots, T$. With the buying price approach and the selling price approach, income streams can also be evaluated by certainty equivalent in Equation (19) and (21), respectively.

For the investors who have preference for consumptions over time and terminal wealth, the optimization problem can be restated as

$$\max_{\substack{\alpha_0^*, \alpha_1^*, \cdots, \alpha_{T-1}^* \\ C_0, C_1, \cdots, C_{T-1}, C_T}} E_0 \left\{ \sum_{t=0}^{T} B_t \left[-\exp\left(-AC_t\right)\right] + B_W \left[-\exp\left(-AW\right)\right] \right\} \tag{39}$$

$$\text{s.t. } W = V_T - C_T$$
$$V_t(\omega) = \left(V_{t-1} - C_{t-1}\right)(1+r) + \alpha_{t-1} X_t(\omega) + H_t(\omega), \forall \omega \in \Omega$$
$$C_t \leq V_t, \ t < T$$

where B_W is the utility weights for terminal wealth.

For this problem, the optimization procedure is somewhat different from the previous one. We need to start from period T, that is

$$J_T = \max_{C_T} \left\{ B_T \left[-\exp\left(-AC_T\right)\right] + B_W \left[-\exp\left(-AV_T + AC_T\right)\right] \right\} \tag{40}$$

Taking the first order derivative with respect to C_T, we have the optimal terminal consumption as follows:

$$C_T = \gamma_T^* + \lambda_T V_T \tag{41}$$

where $\gamma_T^* = -\frac{\ln\left(\frac{B_W}{B_T}\right)}{2A}$ and $\lambda_T = \frac{1}{2}$.

The rest of recursive structures are exactly the same as that in the previous problem.

To interpret the recursive restructures, γ_t can be seen as autonomous consumption, which is the minimum amount that the investor has to consume at time period t. This interpretation of γ_t makes λ_t marginal propensity to consume, which determines marginal consumption for one additional unit of V_t. It is also obvious that λ_t is not affected by income streams while γ_t, on the other hand, is. Consequently, income streams of an investment project do not change the investor's marginal consumptions in exponential utility. However, autonomous consumptions as well as expected utility increases with the amount of income streams.

In addition, as we conduct the recursive optimizations, β_t becomes the new utility weight in place of B_t. Meanwhile, K_t describes the marginal effect on the expected utility due to the optimal trading strategy and income streams. Thus, $-ln(K_t)$ represents the marginal wealth due to hedging income streams at time t.

4.2. **Measuring Market Incompleteness.** We can extend the optimization problem to explore the situation in that income streams are to some extent correlated with security prices. Let S^* denote the vector of payoffs generated by the hedged portfolio of securities under the optimal trading strategies. Suppose we know the distribution of securities prices such that σ_S^2 is known. Let $corr(S^*, H) = \rho_{S^*H}$ denote the correlation coefficient between the payoffs of the hedged portfolio, S^*, and income streams, H. It's assumed that there exists a random variable Y which has no correlation with S^*, i.e., $\sigma_{S^*Y} = 0$. In this situation, income streams can be expressed as a linear combination of S^* and Y as follows:

$$H = W_{S^*} S^* + W_Y Y \tag{42}$$

where W_{S^*} and W_Y are the weights corresponding to S^* and Y, respectively, yet to be solved.

With the assumption of $\sigma_{S^*Y} = 0$, it is easy to obtain the following two equations:

$$\sigma_{S^*H} = W_{S^*} \sigma_{S^*}^2 \tag{43}$$

$$\sigma_H^2 = W_{S^*}^2 \sigma_{S^*}^2 + W_Y^2 \sigma_Y^2 \tag{44}$$

Substituting Equation (43) into $\rho_{S^*H} = \sigma_{S^*H}/\sigma_{S^*}\sigma_H$ and solving for W_{S^*}, we have W_{S^*} expressed by ρ_{S^*H}:

$$W_{S^*} = \frac{\sigma_H \rho_{S^*H}}{\sigma_{S^*}} \tag{45}$$

Substituting Equation (45) back to Equation (44) and solving for W_Y, we also have W_Y expressed by ρ_{S^*H}:

$$W_Y = \frac{\sigma_H \sqrt{1-\rho^2_{S^*H}}}{\sigma_Y} \tag{46}$$

The coefficient of correlation between the income streams and the hedged portfolio can play a crucial role in measuring the degree of market incompleteness. It is easy to observe that as S^* and H are more positively correlated, i.e. $\rho_{S^*H} \to 1$, W_S becomes larger and W_Y becomes smaller. We may interpret the second term in Equation (42), $W_Y Y$, as the shocks leading to market incompleteness. Apparently, the higher coefficient of correlation is, the lower the weights of the shocks terms are. In the case of perfect correlation ($\rho_{S^*H} = 1$), it is convenient to see $W_{S^*} = \sigma_H/\sigma_{S^*}$ and $W_Y = 0$, implying the case of complete markets. Since there are no replicating residuals to be resolved in utility optimization, the value of certainty equivalent converges to the martingale price, $\tilde{H}_t$.

In the other extreme situation in that there is no correlation between the income streams and securities prices, we have $W_{S^*} = 0$ and $W_Y = 1$, suggesting that the overall shocks become substantially large. This also means that these substantial shocks need to be resolved in utility optimization, leading to a variation of certainty equivalent subject to risk attitude. Therefore, certainty equivalent falls into an interval of values which are bounded by investor's risk attitude.

Now, the next logical question is to ask how to construct a random variable, Y, which has no correlation with S^*, i.e.,$\rho_{S^*Y} = \sigma_{S^*Y} = 0$. Note that $\sigma_{S^*Y} = 0$also means $E(S^*Y) - E(S^*)E(Y) = 0$ or $E(S^*Y) = E(S^*)E(Y)$. Suppose that there are k states of the world, which implies that each of both X and Y takes k values for each period, and recall that $S^*_t(\omega_1), \cdots, S^*_t(\omega_k)$ are determined from the optimal trading strategies. The next step is to construct $k-1$ arbitrary values of Y, $Y_t(\omega_1), \cdots, Y_t(\omega_{k-1})$ such that $Y_t(\omega_k)$ can be chosen to satisfy $E(S^*Y) = E(S^*)E(Y)$. The formula of $Y_t(\omega_k)$ is given as follows:

$$Y_t(\omega_k) = \frac{\bar{S}^*_t \left\{\sum_{i=1}^{k-1} P\left[Y_t(\omega_i)\right] Y_t(\omega_i)\right\} - \left\{\sum_{i=1}^{k-1} P\left[S^*_t(\omega_i) Y_t(\omega_i)\right] S^*_t(\omega_i) Y_t(\omega_i)\right\}}{\left\{P\left[S^*_t(\omega_k) Y_t(\omega_k)\right] S^*_t(\omega_k) - P\left[Y_t(\omega_k)\right] \bar{S}^*_t\right\}} \tag{47}$$

5. Numerical Example

5.1. **Classic Investment Problem.** Suppose a firm is offered an investment opportunity to build a plant, assuming that there are no delivery lags. The investment cost now would be \$104 and the payoff will be collected a year later depending on market uncertainty (good or bad). Alternatively, the firm could buy a one-year license to defer the project until more information is available. After careful consideration, the firm feels that currently they have three alternatives to choose from: to invest now, to defer one year, or to decline. There is a diagram of the decision tree of the investment problem associated with the conditional probabilities and payoffs in Figure 2. Suppose, for the purpose of this example, that there are only two securities in the market: the risk-free bond with a return of 8%, and the risky stock with payoffs as shown in Figure 3.

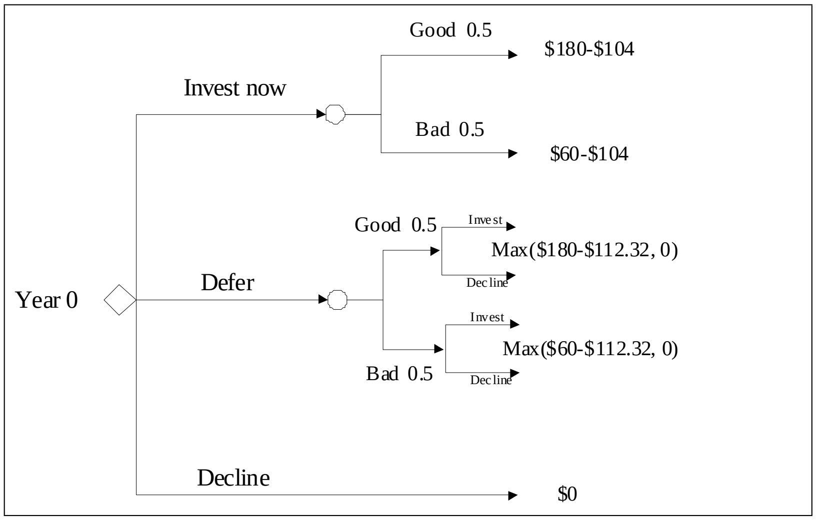

Figure 2. A Classic Investment Problem in Complete Markets

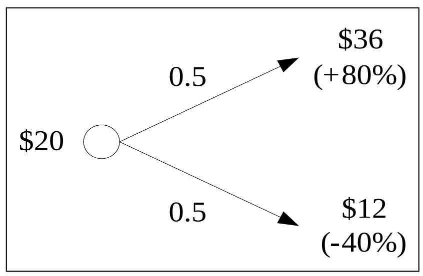

Figure 3. The Stock Dynamics for One Period

This single-period problem was first raised by Trigeorgis and Mason (1987), and then further discussed by Copeland and Weiner (1990) and Nau and McCardle (1991) to illustrate the different approaches to valuation. By presenting this problem, we can review how the different approaches are adopted to solve this problem. Copeland and Weiner (1990) use this problem to make a comparison of the differences between decision tree analysis and contingent claims analysis, and argue that the latter is superior to the former for two reasons: firstly, that the risk-adjusted discount rate is often misused by management, and, secondly, that managerial flexibility is generally ignored in decision tree analysis. However, what they fail to point out is that the validity of contingent claims analysis rests on the assumption that the markets are complete.

On the other hand, Nau and McCardle (1991) argue that the problem may be solved with contingent claims analysis as long as the no-arbitrage principle holds. Thus, the result of a martingale price under the no-arbitrage principle is effectively equivalent to that under the complete markets assumption. Their main departure from conventional real options literature is that optimization techniques are introduced to explicitly model the investor's risk attitude. Nevertheless, Nau and McCardle neglect the reality that most capital investments are mostly non-traded assets on the market, which leads to the presence of arbitrage opportunities. The second disadvantage of their approach is that the investor's fundamental preference is assumed to maximize expected utility over terminal wealth. This assumption may be inappropriate if investors' fundamental preferences are for consumption streams.

The basic investment problem is obviously a real option problem in complete markets since there are two primary securities and two states of the world. Smith and Nau (1995) extend the same investment problem in the situation where the markets are incomplete by recognizing one additional source of uncertainty, operating uncertainty (either efficient or inefficient, for simplicity), as shown in Figure 4. In their framework, they integrate the techniques of utility maximization and decision tree analysis to evaluate the investment opportunity. The specific assumption made for the integrated framework is that project uncertainties are divided into market uncertainty and private uncertainty; the former is assumed to be fully resolved with a risk-neutral probability measure and the latter needs a risk-adjusted probability measure to comply with decision tree analysis. The technique of expected utility maximization is then introduced to consider the investor's risk preference. The specific assumption allows their integrated framework to solve the problem as a special case. Furthermore, in a similar way to Nau and McCardle (1991), Smith and Nau (1995) assume that the objective function is to maximize expected utility over terminal wealth, which may appear to be an inappropriate assumption if the investor's fundamental preference is for consumption. Smith (1998) extends the same integrated framework to maximize expected utility over consumption streams. However, neither Smith and Nau (1995) nor Smith (1998) explicitly solves the recursive structures of deriving a certainty equivalent, which allow us to observe how the value of an investment opportunity in an incomplete market deviates from a martingale and falls into an interval of price bounds.

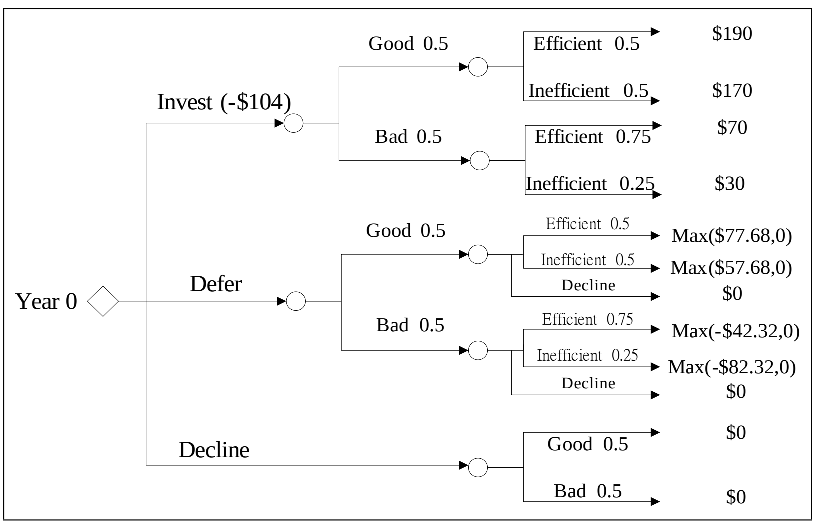

Figure 4. The Extended Investment Problem

5.2. **Problem Solution.** We apply the dynamic programming framework to solve the extended investment problem in the preceding section. Assume that the objective of the management is to maximize expected utility over consumptions and terminal corporate wealth. Since initial wealth is independent of the calculation of certainty equivalent and optimal trading strategies, the firm's initial wealth, V_0, is given as zero for the sake of simplicity. It is also suppose the firm would take out the money equal to the risk-free rate as consumptions.

We then compute the NPVs of all alternatives, i.e., invest now, defer, and decline, with sensitivity to changes in the coefficient of risk aversion. The results are displayed in Table 2 and Figure 5. As shown in Table 2, if the firm decides to launch the investment project immediately at an investment cost of $104, the firm will receive a certainty equivalent value of $83.06, which obviously leads to a negative NPV of -$20.94, assuming the coefficient of risk aversion equal to 1. Also, as the coefficient of risk aversion changes from 1 to 6, the CE value for the "invest now" alternative decreases from 83.06 to 77.01, leading to the changes in NPV from -20.94 to -29.99, i.e., from -20.13% to -25.95% as a percentage of investment cost. The negative relationship between risk aversion and CE is consistent with the derivation in the preceding section.

Table 2. Net Present Values of all Alternatives
(Numbers in brackets are expressed as a percentage of investment cost)

Alternatives		1	2	3	4	5	6
Invest	CE	83.06	82.17	81.03	80.56	78.87	77.01
	NPV	−20.94 (−20.13%)	−21.83 (−20.99%)	−22.97 (−22.09%)	−23.44 (−22.54%)	−25.13 (−24.16%)	−26.99 (−25.95%)
Defer	NPV	20.16 (19.38%)	21.49 (20.66%)	22.04 (21.19%)	22.76 (21.88%)	23.65 (22.74%)	24.87 (23.91%)
Option	Value	41.10 (39.52%)	43.32 (41.65%)	45.01 (43.28%)	46.2 (44.42%)	48.78 (46.90%)	51.86 (49.87%)

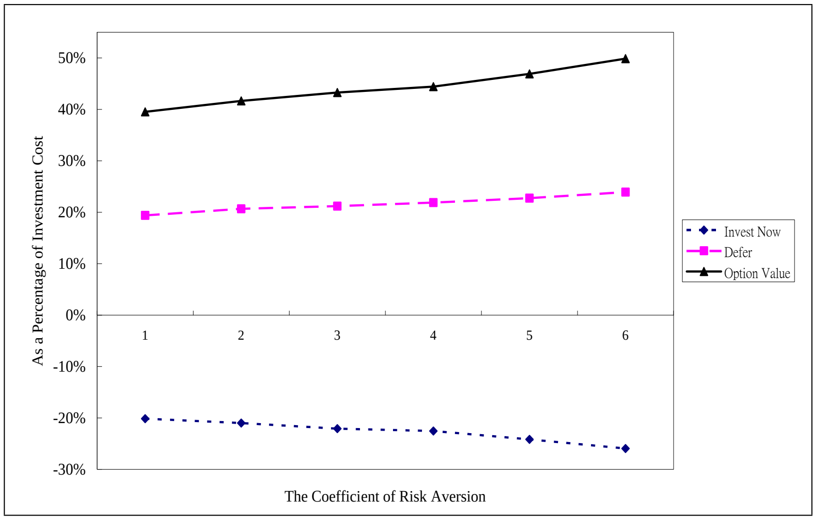

Figure 5. The Values of All Alternatives

For the "defer" alternative, its NPV ranges from 20.16 to 24.87, i.e., from 19.38% to 23.91% as a percentage of investment cost, as risk aversion increases. The option value therefore could be computed by subtracting the NPV of the "invest" alternative. As shown in Table 2, as management becomes more risk-averse, the option value increases from 41.10 to 51.86, i.e., from 39.52% to 49.87% as a percentage of investment cost. Since the "defer" alternative is much more valuable than the other two alternatives regardless of the coefficients of risk aversion, the firm should therefore defer the investment project and wait to see the market conditions one year later, rather than initiate or decline the project immediately.

It is important to point out that there are only two parameters to be determined in our analysis, i.e., initial investment cost and risk-free rate, as the market conditions are given in the classic investment problem. However, since both the CEs and option values are expressed as a percentage of investment cost, the choice of parameter value is out of concern. In addition, as the option to defer does not exist for a long time in the analysis, both the CEs and option values are insensitive to the changes in risk-free rate..

6. Concluding Remarks

In this paper, a dynamic programming framework in discrete time is presented to value an investment project in an incomplete by maximizing expected utility of an investor. Since the equivalent martingale price does not exist in an incomplete market, an investment project must be valued by a certainty equivalent. It is then demonstrated that two approaches to deriving certainty equivalent, the buying price approach and the seller price approach, are exactly equal in the exponential utility, given that the buyer and the seller have same risk preference. This equality implies that the certainty equivalent can be a fair price of income streams for both the buyer and the seller in an incomplete market. Therefore, our approach, compared to alternative models, is relatively intuitive and easy to apply. This model also finds that certainty equivalent tends to increase, as the investment decision-maker becomes more risk-averse. This means that when a decision-maker is more risk-averse, he/she requires more compensations to make up his/her risk tolerance.

In the utility maximization models, this study also develops numerical techniques in order to resolve income streams by dividing them into two components as follows: the hedged positions under the optimal trading strategies and the replicating residuals, the former representing the positions that can be hedged with existing traded securities, and the latter being the residuals that cannot be hedged away. Since the replicating residuals disappear in complete markets, the certainty equivalent of income streams converges to the value of the hedged position. In other words, the objective probability measure in a complete market becomes the risk-neutral probability measure, which leads the CE value to the martingale price. Hence, the CE value converges in complete markets to the martingale price and diverges in an incomplete market into an interval of prices which are conditional on the replicating residuals, a representation of the degree of market incompleteness. Thus, the model based on utility maximization becomes a general valuation framework which may be applied either in a complete or incomplete market.

For empirical implications, the result of our numerical analysis indicates that management's risk aversion could be positively related to option value in an incomplete market, while option value also depends on the degree of market incompleteness. This finding is consistent with the finding of Serra et al. (2011), in which they provide evidence for farmer's decision related to their CE and risk attitude in an incomplete agricultural market. Furthermore, it is demonstrated that this utility-maximization, dynamic-programming approach could be readily applied to evaluate all investment alternatives available to management. In general, the study finds that the approach provides a major advantage over the traditional DCF approach in the considerations of market completeness and investor's risk aversion. This advantage could enhance management making an appropriate investment decision, which is especially important for a near-zero-NPV project.

References

[1] Aiyagari, S. Rao. (1993). Explaining Financial Market Facts: the Importance of Incomplete Markets and Trans-action Costs..Federal Reserve Bank of Minneapolis Quarterly Review 17, 1, 17-31.

[2] Aiyagari, S. R., & Gertler, M. (1991). Asset Returns with Transactions Costs and Uninsured Individual Risk..Journal of Monetary Economics 27, 311-331.

[3] Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities..Journal of Political Economy 81, 637-659.

[4] Bliss, R., & Panigirtzoglou, N. (2004). Option Implied Risk Aversion Estimates..Journal of Finance 59, 1, 407-446.

[5] Brennan, M. J., & Schwartz, E. S. (1982). Consistent Regulatory Policy under Uncertainty.. Bell Journal of Economics 13, 2, 506-521.

[6] Cerny, A., & Hodges, S. (2000). The Theory of Good-Deal Pricing in Financial Markets.. Selected Proceedings of the First Bachelier Conference, Springer.

[7] Cochrane, J., & Saa-Requejo, J. (2000). Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets..Journal of Political Economy 108, 1, 79-119.

[8] Copeland, T., & Antikarov, V. (2001). Real Options: a Practitioner.s Guide. Texere, New York.

[9] Copeland, T., Koller, T., & Murrin, J. (2000). Valuation: Measuring and Managing the Value of Companies. McKinsey & Company, Inc., New York.

[10] Copeland, T., & Weiner, J. (1990). Proactive Management of Uncertainty..McKinsey Quarterly 4, 133-152.

[11] Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). An Intertemporal, General Equilibrium Model of Asset Prices. Econometrica 53, 2, 363-384.

[12] Cox, J., Ross, S., & Rubinstein, M., (1979). Option Pricing: A Simplifed Approach.. Journal of Financial Economics 7, 229-263.

[13] Davis, R. E. (2001). Decision Policy Optimization via Certainty Equivalent Functions for Exponential Utility.Advances in Mathematical Programming and Financial Planning, 6, 89-113.

[14] Duffie, D., & Huang, C. (1985). Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities.Econometrica 53, 6, 1337-1356.

[15] El-Karoui, N., & Quenez, M.C. (1995). Dynamic Programming and Pricing of Contingent Claims in Incomplete Markets..SIAM Journal on Control and Optimization 33, 1, 29-66.

[16] Figlewski, S. (1998) Derivatives Risks, Old and New..Brookings-Wharton Papers on Financial Services 1, 159-217.

[17] Follmer, H., & Leukert, P. (1999). Quantile Hedging..Finance and Stochastics 3, 251-273.

[18] Fugazza, C., Guidolin, M., & Nicodano, G (2009). Time and Risk Diversi.cation in Real Estate Investments: Assessing the Ex Post Economic Value..American Real Estate and Urban Economic Association 37, 3, 341-375.

[19] Green, T. C., & Figlewski, S. (1999). Market Risk and Model Risk for a Financial Institution Writing Options. Journal of Finance 54, 4, 1465-1499.

[20] Hagstromer, B., Anderson, R.G., Binner, J.M., Elger, T., & Nilsson, B. (2008) „Mean-Variance versus Full-Scale Optimization: Broad Evidence for the UK".The Manchester School Supplement, 134-156.

[21] Harrison, J. M., & Kreps, D. M. (1979). Martingales and Arbitrage in Multiperiod Securities Markets. Journal of Economic Theory 20, 381-408.

[22] Henderson, V. (2005). Explicit Solutions to an Optimal Portfolio Choice Problem with Stochastic Income. Journal of Economic Dynamics & Control 29, 7, 1237-1266.

[23] Howard, R. A. (1971). Proximal Decision Analysis..Management Science 17, 507-541.

[24] Hugonnier, J., & Morellec, E. (2007). Corporate Control and Real Investment in Incomplete Markets, Journal of Economic Dynamics and Control 31, 1781-1800.

[25] Kreps, D. (1981). Arbitrage and Equilibrium in Economies with In.nitely Many Commodities.. Journal of Mathematical Economics 8, 15-35.

[26] Jacque, L. (1980). An Expected Utility Framework for Foreign Exchange Decisions: the Long and Short of it..Ph.D. Dissertation, University of Pennsylvania.

[27] Luenberger, D. (1988). Investment Science, Ch. 16. Oxford University Press, Oxford, UK.

[28] Merton, R. (1973). The Theory of Rational Option Pricing.. Bell Journal of Economics and Management Science, 141-183.

[29] Merton, R. (1992). Continuous Time Finance, Blackwell Publishing Co., Cambridge, MA.

[30] Nau, R. F., & McCardle, K.F. (1991).Arbitrage, Rationality, and Equilibrium.. Theory and Decision 31, 199-240.

[31] O'Brien, J.R. (1992). Experimental Stocks with Controlled Risk Preferences,.Journal of Accounting, Auditing, and Finance 7, 2, 117-134.

[32] Pliska, S. R. (1997). Introduction to Mathematical Finance. Blackwell, MA, USA.

[33] Pyo, U. (2008). Real Option Pricing and Bounds in Incomplete Markets..Engineering Economist 53, 4-41.

[34] Rendleman, R.J., & Bartter, B.J. (1979). Two-State Option Pricing.. Journal of Finance 34, 117-134.

[35] Serra, T, Goodwin, B.K., & Featherstone, A.M. (2011). Risk Behavior in the presence of Government Programs..Journal of Econometrics 162, 1, 18-24.

[36] Sick, G.(1995). Real Options.. Handbooks in Operations Research and Management Science. Finance 9, Ch. 21, North-Holland.

[37] Silvers, J. B. (1973) .An Alternative to the Yield Spread as a Measure of Risk..Journal of Finance 28, 4, 933-955.

[38] Smith, J. E. (1998). Evaluating Income Streams: A Decision Analysis Approach.. Management Science 44, 1690-1708.

[39] Smith, J.E., & Nau, R.F. (1995) .Evaluating Risky Projects: Option Pricing Theory and Decision Analysis..Management Science 41, 795-816.

[40] Staum, J. (2004). Fundamental Theorems of Asset Pricing for Good Deal Bounds,.Mathematical Finance, 14, 141-161.

[41] Teisberg, E.O. (1995). Methods for Evaluating Capital Investment Decisions under Uncertainty.. Real Options in Capital Investment, edited by Lenos Trigeorgis, Ch. 2, Praeger, CT.

[42] Trigeorgis, L., & Mason, S.P. (1987). Valuing Managerial Flexibility..Midland Corporate Finance Journal 5, 14-21.

[43] Valenzuela, J., & Mazumdar, M. (2003). Commitment of Electric Power Generators under Stochastic Market Prices..Operations Research 51, 6, 880-892.

[44] Wang, G.Y. (2010). A Dynamic Programming Approach to Real Option Valuation in Incomplete Markets. Proceedings of International Conference of Business Intelligence and Financial Engineering.