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HOW TO COMPUTE THE LIQUIDITY COST IN THE ORDERS-DRIVEN MARKET?

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Abstract. The paper proposes a new method based on stochastic processes theory in order to analyze the equilibrium on the financial markets under asymmetrical information. The paper proposes an analytical formula for the liquidity cost in the orders-driven market taking into consideration the presence of the informed and uninformed investors on the market. This formula is obtained taking into the consideration the fact that an investor who places a limit order offers an option to the rest of the market which can be exercised against him.

1. Introduction

Our particular focus in this paper is on following question. How to compute the liquidity cost in a market governed by orders under asymmetrical information? The answer of this question is related to another questions. First, how do informed and liquidity traders differ in their provision and use of market liquidity? Second, how do characteristics of the market, such as depth in the book or time left to trade, affect these strategies? And, third, how do characteristics of the underlying asset such as asset value volatility affect the provision of market liquidity? Numerous authors in finance have examined aspects of these questions both theoretically and empirically.

The choice between market orders and limit orders has been analyzed in various contexts, see, e.g., Cohen, Maier, Schwartz and Whitcomb (1981), Kumar and Seppi (1994), Chakravarty and Holden (1995), Handa and Schwartz (1996). Dynamic models of order-driven markets include Parlour (1998), Foucault (1999), Foucault, Kadan and Kandel (2005). The price behavior in limit order books has been analyzed theoretically by O'Hara and Oldfield (1986), Rock (1990), Glosten (1994), Seppi (1997) and Biais, Martimort and Rochet (2000). Models that analyze liquidity traders, the dynamics of prices and trades and the convergence of prices to the fundamental value include Glosten and Milgrom (1985), Kyle (1985), Easley and O'Hara (1987), Admati and Pfleiderer (1988). Empirical studies of specific limit order markets include Biais, Hillion and Spatt (1995), Ahn, Bae and Chan (2001), Hasbrouck and Saar (2001), Hollifield, Miller and Sandas (2004) and Hollifield, Miller, Sandas and Slive (2006).

The theory of the financial market microstructure explains the price behavior in limit order book. Generally, in the related literature, the game theory as technical tool was employed. The disadvantage of this technique is the lack of a tractable quantitative measure.

The aim of this paper is to give an answer to the above questions by means of a quantitative measure based on a new technique. This technique derived from the options theory and stochastic processes theory.

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The financial markets microstructure theory can be adapted in order to analyze the market equilibrium using time component. Contrary to the related literature, this paper proposes a new method by means of the stochastic calculus as technical tool in order to describe the equilibrium on the financial markets taking into consideration their microstructure. Therefore, the transaction price (i.e. observable price), S, is considered to be stochastic and to follow a geometric Brownian motion defined by

$$dS_t = \mu S_t dt + \sigma S_t dB_t,\ S_0 = S \tag{1.1}$$

where μ, σ, t and B_t represent the instantaneous return of the financial asset, its volatility, time and a standard Brownian motion. Moreover, the following fact is taken into consideration: on the market, two kind of investors exist, i.e. informed and uninformed investors. The informed investors know the differences between the equilibrium price and the transaction price (i.e. the private information). The private information (denoted by I) is defined by a martingale

$$dI_t = \sigma_i dW_t,\ I_0 = 0 \tag{1.2}$$

where σ_i represents the volatility of the private informations or private signals. The parameters μ, σ and σ_i are constants in time, and the standard Brownian motions, B_t and W_t, are instantaneous correlated, $dW_t dB_t = \rho dt$. The parameter ρ is constant in time and represents the instantaneous correlation coefficient. On the market, a positive signal indicates that the price goes up and a negative signal indicates that the price goes down. The informed traders make profit exploiting these signals. Their trades lead the market to go up or down as quickly as it can. Consequently, the correlation coefficient is set to be positive, $\rho > 0$. If $\rho = 0$, the private signals and the trading prices are not correlated and the informed traders strategies have no impact on the transaction prices. In this case, the informed traders perfectly hide their presence on the market. Obviously, it is complete unrealistic to consider $\rho < 0$ because the informed trader will lose in spite of his information advantage. Thus, the most realistic hypothesis which can be made is $\rho \geq 0$.

Under these hypotheses, the equilibrium price, denoted by P, is defined by

$$P_t = S_t + I_t,\ P_0 = P \tag{1.3}$$

At the current date,

$$P = S \tag{1.4}$$

the equilibrium price is the transaction price. For any arbitrary future date, the equilibrium price is equal to the transaction price plus the value due to the transactions made by informed investors. In the situation in which informed investors don't exist on the market and the information is entirely public then the equilibrium price would be equal to the transaction price. However, in time, the transactions made by the informed investors will be discovered by the uniformed investors and consequently the transaction price tends to the equilibrium price. For this reason, the equilibrium price is equal to the transaction price at any current date. From this date, the uninformed traders observe only the transaction price, while the informed traders observe the transaction price and their private information.

The paper is organized as follows: section 2 analyses the equilibrium price of the financial asset in the stochastic environment, section 3 presents the derivation of a liquidity cost formula when the market is driven by orders, section 4 shows the empirical results and section 5 summarizes and concludes.

2. Equilibrium Price

First of all, the stochastic characterization of the market equilibrium is necessary to be defined. The stochastic dynamic of the equilibrium price is given by

$$dP_t = dS_t + dI_t \tag{2.1}$$

Using the definitions of the stochastic processes followed by S and I, the equilibrium price is defined by the following stochastic differential equation:

$$dP_t = \mu S_t dt + \sigma S_t dB_t + \sigma_i dW_t \tag{2.2}$$

In dynamic, the equilibrium price is continuously adjusted with the private information through the trading made by the informed trader. Taking advantage from the private information, the informed agents will eliminate the risks of their trading strategies. At the market equilibrium, while there is no arbitrage opportunity, the informed traders portfolios can be considered as risk-neutral portfolios. For this reason, is not unrealistic to take into consideration the hypothesis that the informed traders are risk-neutral. When the informed traders are risk-neutral and there are no arbitrage opportunities, they will adjust the equilibrium price with their trading strategies such as the present value of the equilibrium price is a martingale under a risk-neutral probability measure. Let consider the function $e^{-rt}P_t$, where r is the risk-free interest rate. By Itô's lemma

$$d\left(e^{-rt}P_t\right) = \left(\mu S_t e^{-rt} - rP_t e^{-rt}\right) dt + \sigma S_t e^{-rt} dB_t + \sigma_i e^{-rt} dW_t \tag{2.3}$$

Taking into consideration the equation (1.3), the following dynamic is obtained

$$\begin{aligned} d\left(e^{-rt}P_t\right) &= \left[(\mu - r) S_t e^{-rt} - rI_t\right] dt + \sigma S_t e^{-rt} dB_t + \sigma_i e^{-rt} dW_t \\ &= e^{-rt}\sigma S_t \left(dB_t + \frac{\mu - r}{\sigma} dt\right) + e^{-rt}\sigma_i \left(dW_t - \frac{rI_t}{\sigma_i} dt\right) \\ &= e^{-rt}\sigma S_t dB_t^* + e^{-rt}\sigma_i dW_t^* \end{aligned} \tag{2.4}$$

where B_t^*, W_t^* are the Brownian motions under a risk-neutral probability measure defined by means of Girsanov transformation[1]:

$$dB_t^* = dB_t + \lambda_s(t)\, dt = dB_t + \frac{\mu - r}{\sigma} dt \tag{2.5}$$

$$dW_t^* = dW_t + \lambda_i(t)\, dt = dW_t - \frac{rI_t}{\sigma_i} dt \tag{2.6}$$

where $\lambda_s(t) = (\mu - r)/\sigma$ is the trading risk premium (or the market price of trading risk) and $\lambda_i(t) = -rI_t/\sigma_i$ is the information risk premium (or the market price of information risk). The trading risk premium is constant and the information risk premium is state-variable dependent. While the trading risk premium is typically positive, the sign of the information risk premium depends on the private information sign. A positive private signal that the price goes up leads to a negative risk premium, while a negative private signal that the price goes down induces a positive risk premium.

Taking into consideration the market price of trading risk $\lambda_s(t)$ and the market price of informational risk $\lambda_i(t)$, the stochastic dynamics of the trading price and the private information are transformed into risk-neutral dynamics. The both risk-neutral dynamics have the adjusted-drift proportional with the free-risk interest rate. Thus, the risk-neutral dynamics of the transaction price and private information which are defined by

$$dS_t = rS_t dt + \sigma S_t dB_t^* \tag{2.7}$$

$$dI_t = rI_t dt + \sigma_i dW_t^* \tag{2.8}$$

Using the Itô's lemma for the functions $\ln S = f(S,t)$ and $e^{-rt}I = f(I,t)$, the following dynamics are obtained

$$d(\ln S_t) = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dB_t^* \tag{2.9}$$

$$d\left(e^{-rt}I_t\right) = e^{-rt}\sigma_i dW_t^* \tag{2.10}$$

[1]See Appendix.

Knowing that $S_0 = S$ and $I_0 = 0$, the solutions are given by

$$S_T = S \exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \sigma B_T^*\right] \tag{2.11}$$

$$I_T = e^{rT} \int_0^T e^{-rt} \sigma_i dW_t^* \tag{2.12}$$

The obtained relations show that, in a risk-neutral market, the trading price has a log-normal probability distribution and the private information has a normal probability distribution. From these expressions, the mean and the variance for each variable can be determined. Thus,

$$S_T \sim LN\left(Se^{rT}, S^2 e^{2rT}\left(e^{\sigma^2 T} - 1\right)\right) \tag{2.13}$$

$$I_T \sim N\left(0, \frac{\sigma_i^2}{2r}\left(e^{2rT} - 1\right)\right) \tag{2.14}$$

where

$$E_Q\left[S_T|S\right] = Se^{rT} \tag{2.15}$$

$$VAR_Q\left[S_T|S\right] = S^2 e^{2rT}\left(e^{\sigma^2 T} - 1\right) \tag{2.16}$$

$$E_Q\left[I_T|I\right] = 0 \tag{2.17}$$

$$VAR_Q\left[I_T|I\right] = \frac{\sigma_i^2}{2r}\left(e^{2rT} - 1\right) \tag{2.18}$$

2.1. **The Expected Value of the Equilibrium Price.** Taking into consideration the risk-neutral dynamics of the trading price and private information, the equilibrium price has the following risk-neutral dynamic

$$dP_t = rP_t dt + \sigma S_t dB_t^* + \sigma_i dW_t^* \tag{2.19}$$

where $P_0 = P = S$. Using the equation (2.4), the solution of the above stochastic differential equation is given by

$$P_T = Pe^{rT} + e^{rT} \int_0^T e^{-rt} \sigma S_t dB_t^* + e^{rT} \int_0^T e^{-rt} \sigma_i dB_t^* \tag{2.20}$$

Thus, the expected equilibrium price will be defined by[2]

$$E_Q\left[P_T|S\right] = Pe^{rT} = Se^{rT} \tag{2.21}$$

under a risk-neutral probability measure Q. Taking into consideration the expected trading price formula, in a market with asymmetrical information,

$$E_Q\left[P_T|S\right] = E_Q\left[S_T|S\right] \tag{2.22}$$

Alternatively,

$$E_Q\left[P_T|S\right] = E_Q\left[S_T + I_T|S\right] = E_Q\left[S_T|S\right] + E_Q\left[I_T|S\right] = E_Q\left[S_T|S\right] = Se^{rT} \tag{2.23}$$

Under asymmetrical information, the expected value of the equilibrium price is equal with the expected value of the transaction price. This result is intuitively correct because it proves a fundamental reason of market mechanism: all the investors expect that the equilibrium price is the trading price.

The probability distribution of the equilibrium price is the result of the combination between a normal variable and a log-normal variable. The private information could explain the departure from normality of the asset returns.

The variance of the equilibrium price is given by

$$\begin{aligned} VAR_Q\left[P_T|S\right] &= VAR_Q\left[S_T + I_T|S\right] = VAR_Q\left[S_T|S\right] + VAR_Q\left[I_T|S\right] + \\ &\quad 2\rho\sqrt{VAR_Q\left[S_T|S\right] VAR_Q\left[I_T|S\right]} \end{aligned} \tag{2.24}$$

[2]Knowing that $I = 0$ and $P = S$, $E_Q\left[P_T|P\right] = E_Q\left[P_T|S, I\right]$ is written as $E_Q\left[P_T|S\right]$.

$$\begin{aligned} VAR_Q\left[P_T|S\right] &= S^2e^{2rT}\left(e^{\sigma^2T}-1\right)+\frac{\sigma_i^2}{2r}\left(e^{2rT}-1\right)+ \\ &\quad 2\sigma_iSe^{rT}\frac{\rho}{\sqrt{2r}}\left[\left(e^{\sigma^2T}-1\right)\left(e^{2rT}-1\right)\right]^{1/2} \end{aligned} \tag{2.25}$$

Using the approximation $e^x \simeq 1+x$, for x small, the equilibrium price variance is

$$\begin{aligned} VAR_Q\left[P_T|S\right] &= S^2\left(1+2rT\right)\sigma^2T+\sigma_i^2T+2\rho\sigma_i\sigma S\left(1+rT\right)T \\ &= VAR_Q\left[S_T|S\right]+\sigma_i^2T+2\rho\sigma_i\sigma S\left(1+rT\right)T \end{aligned} \tag{2.26}$$

The informational efficiency of the market is defined by the ratio between the trading price variance and the equilibrium price variance. Letting

$$Z=\frac{VAR_Q\left[S_T|S\right]}{VAR_Q\left[P_T|S\right]} \tag{2.27}$$

Z is the market informational efficiency. Because $\rho \geq 0$, $VAR_Q\left[P_T|S\right] > VAR_Q\left[S_T|S\right]$ and, consequently, the ratio Z varies between 0 and 1. The ratio Z represents the weight of the equilibrium price variance explained by the trading price variance. In other words, Z explains how much the trading price variability is due to the private signals and how much this information is included in trading price. If Z is close to 1, the market has a higher informational efficiency and private signals are easily discovered by the uninformed traders. If Z is close to 0, the market informational efficiency is lower and the private signals are hardly discovered by the uniformed investors. If $Z=1$, the private informations are revealed to all investors, informed or uninformed. In this case, the equilibrium is perfectly revealed on the market. The market informational efficiency is defined by

$$Z=\frac{S^2\left(1+2rT\right)\sigma^2}{S^2\left(1+2rT\right)\sigma^2+\sigma_i^2+2\rho\sigma_i\sigma S\left(1+rT\right)}<1 \tag{2.28}$$

If $\rho=0$, the market informational efficiency is

$$Z=\frac{S^2\left(1+2rT\right)\sigma^2}{S^2\left(1+2rT\right)\sigma^2+\sigma_i^2}<1 \tag{2.29}$$

In an ideal case, the market is perfectly informational efficient ($Z=1$) when $VAR_Q\left[P_T|S\right]=VAR_Q\left[S_T|S\right]$. In this case, knowing that $\rho \geq 0$,

$$\sigma_i^2+2\rho\sigma_i\sigma S\left(1+rT\right)=0, \text{ which is equivalent with } \sigma_i=0$$

The market tends to the perfect informational efficiency when the private information (signal) variability is very low ($\sigma_i \to 0$).

3. Liquidity Cost

Taking into consideration the market microstructure, the liquidity has two alternative sources: prices negotiated by the market makers, if the market is driven by prices, or prices negotiated by the final investors, if the market is driven by orders. On the continuous market, a limit order is risky because its execution depends on the market conditions changes. Let's consider a situation where an investor places a selling limit order at 100 €. If a new information arrives on market justifying a new price at 101 € and the investor is not willing to quickly change the order, then other investors would have the opportunity to gain 1 €. This phenomenon can be described using the option theory: the investor who places a limit order offers an option to the rest of the market which can be exercised against him if the market goes contrary.

On the one hand, in the auction theory, the winner of an auction overestimates the value of the object to sell. Hence, the winner is "cursed" to pay a higher price. The investor who places a limit order is faced with a similar problem. Due to its optional character, a buying limit order risks to be executed only if the real value of the asset becomes lower than the offered price (which means that the price overestimates the real value of the asset). Similarly, a selling limit order risks to be executed only if the price underestimates the real value of the asset.

On the other hand, the risk of a limit order can be explained by information asymmetry. Thus, an investor who places a limit order is faced with the adverse selection risk. For example, a buying limit order allows an informed investor who knows that the real value of the asset is lower than the offered price to take advantage from his information against the buyer who gives the limit order.

Therefore, the investor will be less incited to place the limit orders and the market liquidity will decreases. This risk appears especially on markets with the automated execution of orders. Hereby, the market can quickly profit from the selling or buying limit orders which overestimate or underestimate the value of the financial assets before the investors have time to cancel or modify the limit orders.

From now on, the option theory in order to obtain a formula of the liquidity cost is used. Therefore, the market is considered to be a continuous market driven by orders with unlimited time execution of orders, such as French or Japanese stock exchange. A buying limit order gives to other investors the right but not the obligation to sell the financial asset at limit price offered for unlimited time. Therefore, the liquidity cost payable by the investor who gives the limit order is the price of a perpetual American put. The liquidity cost is defined by

$$L = \max_{\tau_l} E_Q \left[e^{-r\tau_l} \left(K - S_{\tau_l} \right) \right] \tag{3.1}$$

where K is the limit price offered by the buying limit order. $E_Q \left[e^{-r\tau_l} \left(K - S_{\tau_l} \right) \right]$ is the expected value under a risk neutral probability, Q, of the option payoff discounted at the risk free interest rate, r. τ_l is a stopping time. In a risk neutral world, the stochastic dynamic of the transaction price is given by

$$dS_t = rS_t dt + \sigma S_t dB_t^* \tag{3.2}$$

Let X a known positive level of the equilibrium price, P, so that $X < K$. If the current transaction price, S, is equal or lower than X, the buying limit order is executed instantly (or the put option is executed instantly). The value of the perpetual American put option will be $K - S$, because $\tau_l = 0$. If the current transaction price, S, is higher than X, the option will be executed at the stopping time τ_l defined by

$$\tau_l = \min \left\{ t \geq 0; S(t) = P(t) = X \right\} \tag{3.3}$$

where τ_l is ∞ if the price of the financial asset never reaches the value X. At exercise time, the value of the put option will be $K - S_{\tau_l} = K - X$. Hereby, the liquidity cost is

$$L = (K - X) E_Q \left[e^{-r\tau_l} \right] \quad \text{for all } S > X \tag{3.4}$$

Using Itô lemma, the solution of the stochastic differential equation (3.2) is given by

$$S(t) = S \exp \left[\sigma B_t^* + \left(r - \frac{\sigma^2}{2} \right) t \right] \tag{3.5}$$

The stopping time τ_l is the moment when the price reaches the level X. But $S(t) = X$, if and only if

$$-B_t^* - \frac{1}{\sigma} \left(r - \frac{\sigma^2}{2} \right) t = \frac{1}{\sigma} \ln \frac{S}{X} \tag{3.6}$$

In order to get $E_Q \left[e^{-r\tau_l} \right]$ the following theorem is used.

Theorem 3.1. *Let W_t^* a standard Brownian motion under the probability Q, let γ a real number and h a positive number. Let the stochastic process*

$$Y(t) = \gamma t + B_t^*$$

and the stopping time

$$\tau_h = \min \left\{ t \geq 0; Y(t) = h \right\}$$

Then[3]

$$E_Q\left[e^{-\lambda\tau_h}\right] = e^{-h\left(-\gamma+\sqrt{\gamma^2+2\lambda}\right)} \quad \textit{for all } \lambda > 0$$

Replacing λ with r, γ with $-\frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)$ and h with $\frac{1}{\sigma}\ln\frac{S}{X}$, the next result is obtained

$$\begin{aligned}
-\gamma+\sqrt{\gamma^2+2\lambda} &= \frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)+\sqrt{\frac{1}{\sigma^2}\left(r-\frac{\sigma^2}{2}\right)^2+2r} \\
&= \frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{\sigma}\sqrt{\left(r+\frac{\sigma^2}{2}\right)^2} \\
&= \frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{\sigma}\left(r+\frac{\sigma^2}{2}\right)=\frac{2r}{\sigma}
\end{aligned}$$

The enunciated theorem implies the following result

$$E_Q\left[e^{-r\tau_l}\right] = \exp\left[-\frac{1}{\sigma}\ln\frac{S}{X}\frac{2r}{\sigma}\right] = \left(\frac{S}{X}\right)^{-\frac{2r}{\sigma^2}} \tag{3.7}$$

Therefore, the liquidity cost payable by an investor who gives a buying limit order at the limit price K is:

$$L = \begin{cases} K-S, & \text{if } 0 \le S \le X \\ (K-X)\left(\frac{S}{X}\right)^{-\frac{2r}{\sigma^2}}, & \text{if } S > X \end{cases} \tag{3.8}$$

Until now, the problem of the liquidity cost is treated for an arbitrary value of the equilibrium price X. From now on, the liquidity cost is examined for an optimum value of X. For S fixed, let X^* the optimum value of X which maximizes the amount:

$$g(X) = (K-X)\,X^{\frac{2r}{\sigma^2}}S^{-\frac{2r}{\sigma^2}} \tag{3.9}$$

Because $\frac{2r}{\sigma^2}$ is strictly positive, $g(0)=0$ and $\lim_{X\to\infty} g(X) = -\infty$. Moreover,

$$g'(X) = S^{-\frac{2r}{\sigma^2}}\left[K\frac{2r}{\sigma^2}X^{\frac{2r}{\sigma^2}-1}-\left(\frac{2r}{\sigma^2}+1\right)X^{\frac{2r}{\sigma^2}}\right] \tag{3.10}$$

Using the first order condition, $g'(X^*)=0$, the following result is obtained

$$K\frac{2r}{\sigma^2}(X^*)^{\frac{2r}{\sigma^2}-1} = \left(\frac{2r}{\sigma^2}+1\right)X^{\frac{2r}{\sigma^2}} \tag{3.11}$$

which implies

$$X^* = \frac{2r}{2r+\sigma^2}K \tag{3.12}$$

The obtained result is a number between 0 and K, that is $X^* < K$. Consequently, the function $g(X^*)$ can be written

$$g(X^*) = \frac{\sigma^2}{2r+\sigma^2}\left(\frac{2r}{2r+\sigma^2}\right)^{\frac{2r}{\sigma^2}}K^{\frac{2r+\sigma^2}{\sigma^2}}S^{-\frac{2r}{\sigma^2}} \tag{3.13}$$

Consequently, in the presence of the informed investors on the market, the final formula of the liquidity cost on a market driven by orders for a limit price K is given by

$$L = \begin{cases} K-S, & \text{if } 0 \le S \le \frac{2r}{2r+\sigma^2}K \\ \frac{\sigma^2}{2r+\sigma^2}\left(\frac{2r}{2r+\sigma^2}\right)^{\frac{2r}{\sigma^2}}K^{\frac{2r+\sigma^2}{\sigma^2}}S^{-\frac{2r}{\sigma^2}}, & \text{if } S > \frac{2r}{2r+\sigma^2}K \end{cases} \tag{3.14}$$

[3]See the proof of the theorem in Shreve, S. (2004), Stochastic Calculus for Finance, Springer, New York, volume II, pages 346-347.

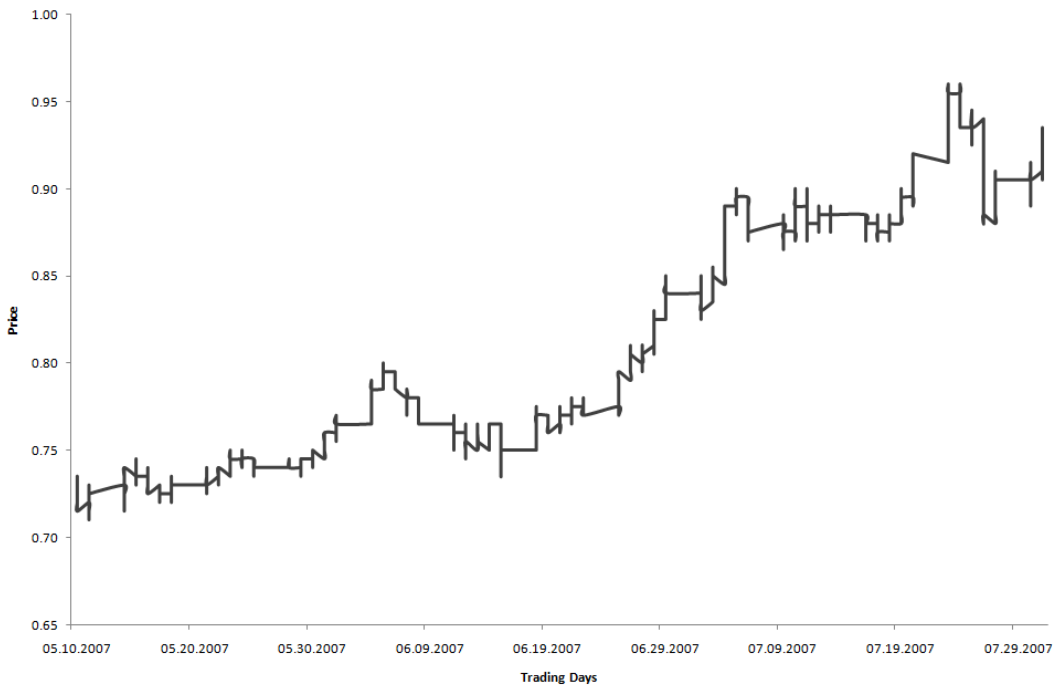
4. Empirical Results

In the literature, the empirical papers include Biais, Hillion, and Spatt (1995), who document the diagonal effect (positive autocorrelation of order flow) and the comovement effect (e.g., a downward move in the bid due to a large sell market order is followed by a smaller downward move in the ask – which increases the bid-ask spread); Hollifield, Miller and Sandas (2004) who test monotonicity conditions resulting from a dynamic model of the limit order book and provides some support for it; Hollifield, Miller, Sandas and Slive (2006) who use data from the Vancouver exchange to find that agents supply liquidity (by limit orders) when it is expensive and demand liquidity (by market orders) when it is cheap. Consequently, a natural question is which is the cost for the agents to supply liquidity by limit orders? A quantitative measure for this liquidity cost is given by relation (3.14).

In this section, the liquidity cost formula is empirically analyzed. A database which includes the intraday transaction prices of the most negotiated securities on the Bucharest Stock Exchange is used. The database contains the intraday transaction prices from May 10, 2007 to July 31, 2007. The sample comprises 7076 records. Also, the database contains the daily interest rates. The average of the daily interest rates of the study period was 0.0187%.

Taking into consideration one of the financial assets with the greatest trading volume (i.e. Banca Transilvania S.A.), its trading prices evolution is shown in Figure 1.

Figure 1. The Intraday Transactions Prices



For every trading day the volatility was computed as standard deviation of the intraday transaction prices. The average of the daily volatility over the period May 10, 2007 - July 31, 2007 was 0.4792%. The Figure 2 shows the daily volatilities of the Banca Transilvania S.A. shares on the study period.

Figure 2. The Daily Volatility

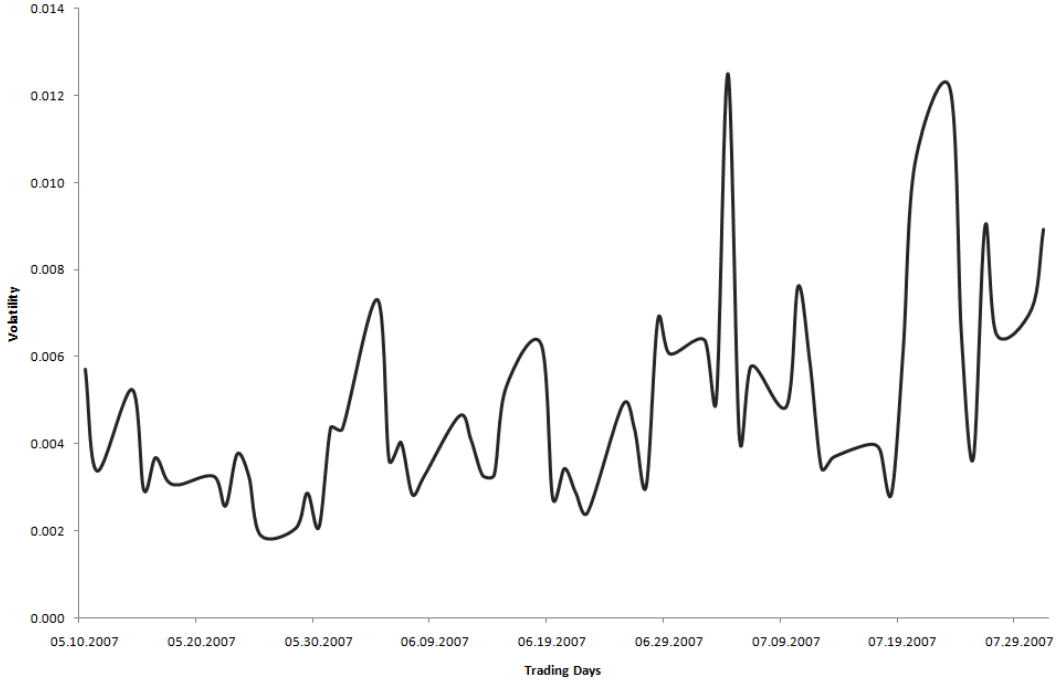


Figure 3. The Mean Liquidity Cost

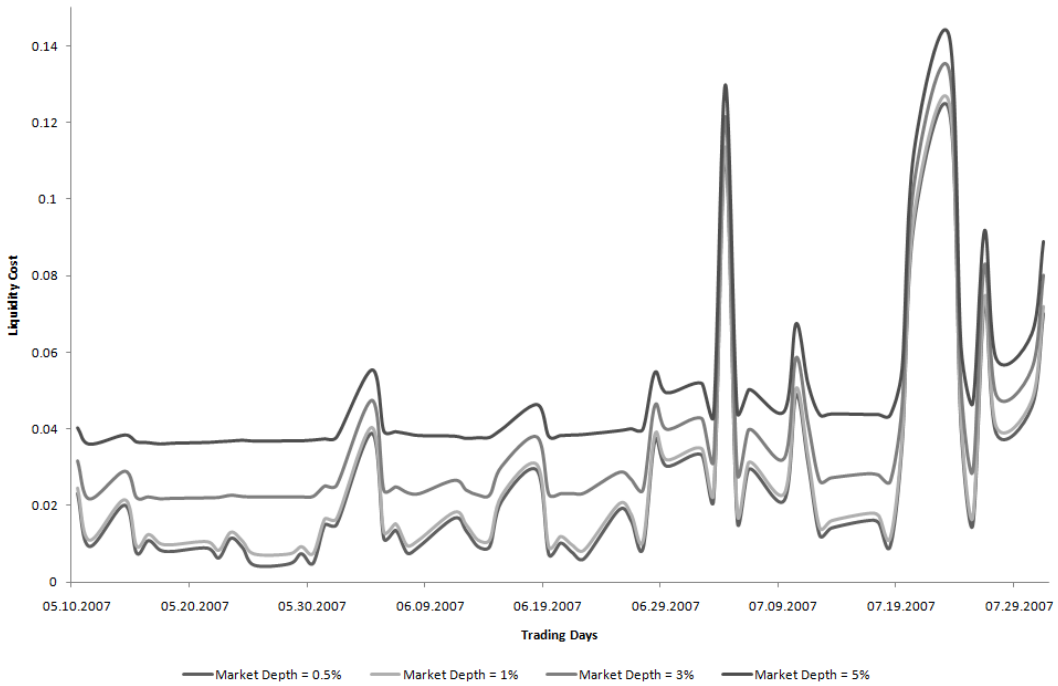


Figure 4. Differences between the Liquidity Costs with 0.5% and 1% Market Depth

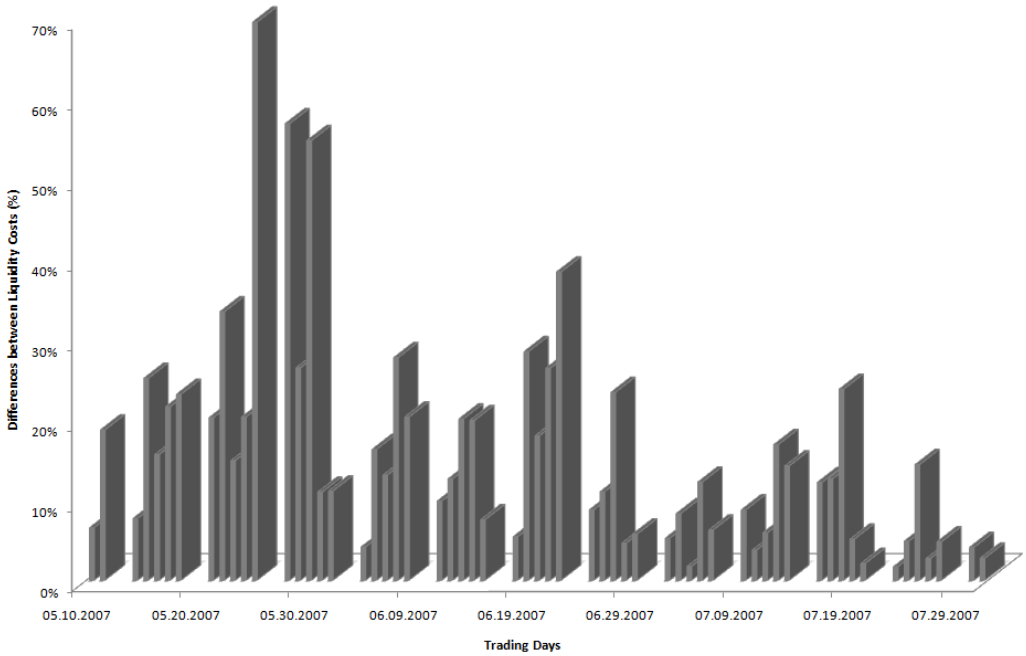
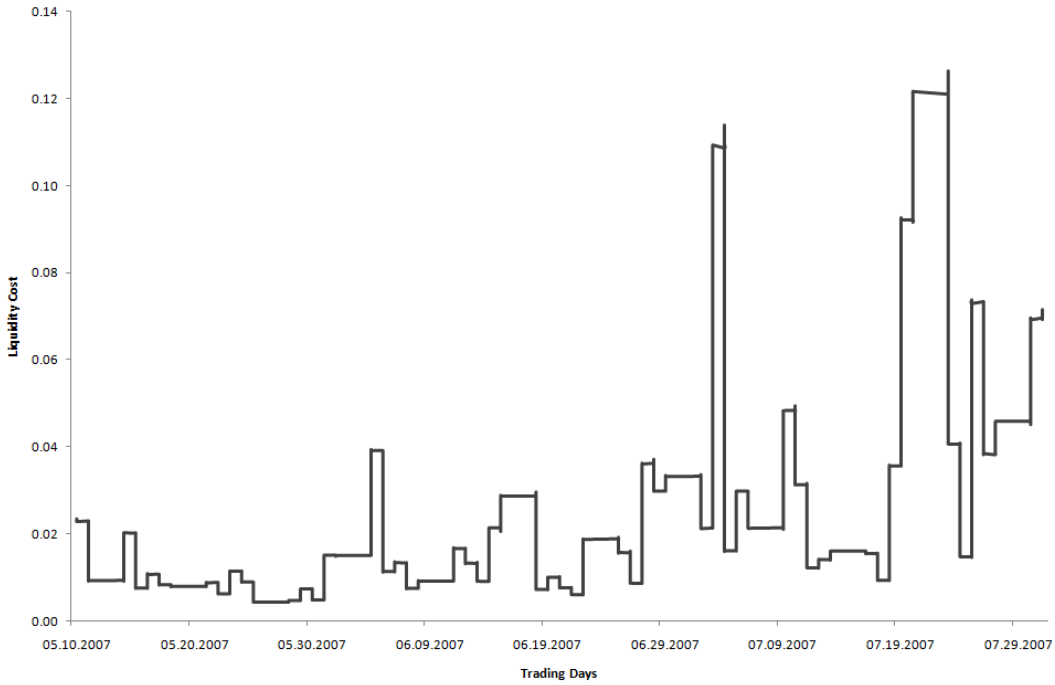


Figure 5. The Intraday Liquidity Cost for 0.5% Market Depth



Using the formula (3.14) of the liquidity cost, the market depth is arbitrary supposed to be equal to 0.5%, 1%, 3% or 5%. The market depth is defined as a percentage variation of the limit price with respect to the transaction price:

$$\text{Market Depth} = \frac{K - S}{S} \times 100 \tag{4.1}$$

The Figure 3 exhibits the evolution of the mean daily liquidity cost for four arbitrary values of the financial asset market depth (i.e. 0.5%, 1%, 3% and 5%). The Figures 3 and 4 compare the liquidity costs for different values of the asset market depth. The conclusion is that an increase of the market depth implies an increase of the liquidity cost on the financial asset market. The differences between the liquidity costs for 1% market depth and for 0.5% market depth are always positives and they vary to 70% maximum.

Table I. Descriptive Statistics for Intraday Liquidity Cost

	Mean	Standard Deviation	Skewness	Kurtosis	Min Value	Max Value	Range	Median
Market Depth 0.5%	0.0256	0.0254	2.2421	7.8139	0.0043	0.1263	0.1219	0.0160
Market Depth 1%	0.0274	0.0254	2.2572	7.8653	0.0073	0.1284	0.1210	0.0177
Market Depth 3%	0.0374	0.0243	2.4434	8.6138	0.0215	0.1371	0.1156	0.0282
Market Depth 5%	0.0500	0.0225	2.6398	9.5571	0.0357	0.1462	0.1104	0.0403

The Figure 5 shows the evolution of the intraday liquidity cost for the study period, from May 10, 2007 to July 31, 2007. The liquidity cost is computed for 0.5% market depth.

The Table I shows the descriptive statistics of the intraday liquidity costs with 0.5%, 1%, 3% and 5% market depth. It can be noticed that the mean value of the liquidity cost varies from 0.0256 euro for 0.5% market depth to 0.0500 euro for 5% market depth. Also, the extreme values increase with the rising market depth. On the other hand, the standard deviation of the liquidity cost decreases with the rising market depth. Concluding, the mean liquidity cost on the financial asset (i.e. Banca Transilvania S.A.) market driven by orders represents about 3% of the transaction prices of the study period.

5. Conclusion

Based on classical hypotheses used in stochastic calculus applied in finance, the paper shows that on a financial market with information asymmetry all investors expect that the equilibrium price is the current transaction price. The paper defines an indicator of the market informational efficiency as the ratio between trading price and equilibrium price variances. Because the equilibrium price variance is higher than the trading price variance, the market informational efficiency indicator is defined between 0 and 1. If the indicator value is close to 1, the market is informational efficient. Using the stochastic dynamics hypotheses, the paper proposes a measurement of the liquidity cost on a market driven by orders. The proposed analytical formula of the liquidity cost of the financial asset market driven by orders depends on four parameters: the risk free interest rate, the transaction price of the financial asset, the volatility of the financial asset and the limit price offered by the buying limit order.

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Appendix

In order to obtain the martingale restriction for the present value of the equilibrium price, the multidimensional Girsanov Theorem is used.

The multidimensional Girsanov Theorem states that

$$M_j^*(t) = M_j(t) + \int_0^t \Theta_j(u)\,du, \quad j = 1,..,d \tag{A1}$$

or

$$M^*(t) = M(t) + \int_0^t \Theta(u)\,du \tag{A2}$$

where $M(t) = (M_1(t),..,M_d(t))$ is a d-dimensional Brownian motion[4] under the actual probability measure P, $M^*(t) = (M_1^*(t),..,M_d^*(t))$ is a d-dimensional Brownian motion under the equivalent probability measure Q, and $\Theta(t) = (\Theta_1(t),..,\Theta_d(t))$ is d-dimensional adaptive process. The component processes of $M(t)$ are independent under P, but each $\Theta_j(t)$ processes can depend in a path-dependent, adapted way on all of the Brownian motions $M_1(t)$,..,$M_d(t)$.

Knowing that the Brownian motions B_t and W_t are correlated ($dB_t dW_t = \rho dt$), in order to apply the multidimensional Girsanov Theorem, the Brownian motion W_t is written as a function of two independent Brownian motions as follows

$$dW_t = \rho dB_t + \sqrt{1-\rho^2}dU_t \tag{A3}$$

where U_t is a Brownian motion. The Brownian motions B_t and U_t are independent. By multidimensional Girsanov Theorem, these independent Brownian motions are "redefined" under an equivalent probability measure

$$dB_t^* = dB_t + \lambda_s(t)\,dt \tag{A4}$$

$$dU_t^* = dU_t + \lambda_u(t)\,dt \tag{A5}$$

where $\lambda_s(t)$ is the trading risk premium and $\lambda_u(t)$ is the risk premium which correspond to the source of risk induced by the Brownian motion U_t. Therefore, the Brownian motion W_t can be defined by

$$\begin{aligned} dW_t &= \rho\left[dB_t^* - \lambda_s(t)\,dt\right] + \sqrt{1-\rho^2}\left[dU_t^* - \lambda_u(t)\,dt\right] \\ &= \left(\rho dB_t^* + \sqrt{1-\rho^2}dU_t^*\right) - \left[\rho\lambda_s(t) + \sqrt{1-\rho^2}\lambda_u(t)\right]dt \end{aligned} \tag{A6}$$

Let W_t^* a Brownian motion such as

$$dW_t^* = \rho dB_t^* + \sqrt{1-\rho^2}dU_t^* \tag{A7}$$

and let $\lambda_i(t)$ a risk premium such as

$$\lambda_i(t) = \rho\lambda_s(t) + \sqrt{1-\rho^2}\lambda_u(t) \tag{A8}$$

Using the above relations, the Brownian motion W_t can be "redefined" under an equivalent probability measure as follows

$$dW_t = dW_t^* - \lambda_i(t)\,dt \quad \text{or} \quad dW_t^* = dW_t + \lambda_i(t)\,dt \tag{A9}$$

Therefore, the Girsanov transformation leads to the following simultaneously relations

$$dB_t^* = dB_t + \lambda_s(t)\,dt \tag{A10}$$

$$dW_t^* = dW_t + \lambda_i(t)\,dt \tag{A11}$$

where B_t^* and W_t^* are the Brownian motions defined under an equivalent probability measure, $\lambda_s(t)$ is the trading risk premium and $\lambda_i(t) = \rho\lambda_s(t) + \sqrt{1-\rho^2}\lambda_u(t)$ is the information risk premium.

[4] A multidimensional Brownian motion is a vector of independent one-dimensional Brownian motions.