The Review of Finance and Banking
Volume 05, Issue 1, Year 2013, Pages 047–061
S print ISSN 2067-2713, online ISSN 2067-3825

# EMPIRICAL COMPARISON OF ROBUST PORTFOLIOS' INVESTMENT EFFECTS

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Abstract. The purpose of this article is to assess whether correct application of robust estimators in construction of minimum variance portfolios' (MVP) allows to achieve better investment results in comparison with portfolios defined using classical MLE estimators. Theoretical robust portfolios properties and portfolios investment effect are compared.
This paper proposes a practical methodology of comparing alternative estimation methods, based on random portfolio selection. This approach enables to analyse investment effects of various portfolios.
The empirical analysis shows that for MVP portfolios with nonnegative constraints created using robust methods, there is no significant risk improvement, and that even for most robust methods, there is an observable significant risk increase compared to the risk of classical portfolios. This paper also shows that robust portfolio estimators cause higher transaction cost.

## 1. Introduction

The purpose of this article is to assess whether correct application of robust estimators in construction of minimum variance portfolios' (MVP) allows to achieve better investment results (measured in terms of portfolio risk and return) in comparison with portfolios defined using classical estimators.
This paper proposes a methodology of comparison of portfolios investment performance (risk, return, turnover) consisting in comparing the performance of $N$ portfolios which contain $p$ random assets from an investigated set of $M$ assets ($M > p$). The weights of each of the $N$ portfolios are determined using classical estimators (MLE), robust estimators and other non-classical estimators. As opposed to the research to date, which employed historical data, this comparison is not being made on the basis of a single portfolio performance ($N = 1$) consisting chosen group of assets, but on the basis of the performance of $N$ ($N > 1$) portfolios. Such an approach will allow to compare performance not only for single portfolio, but also for many portfolios, which will enable to examine the effectiveness of the investigated methods for any portfolio generated by an investor.
The methodology has the hidden assumption that the investors' portfolios contain an initial set of stocks determined on the basis of technical and fundamental analysis, investors' preferences or risk aversion. The Markowitz approach to minimize the risk is applied to this given portfolio as opposed to the case in which optimal portfolio weights are estimated from the entire set of assets.
Using the previously described approach, this article makes performance comparison between classical portfolios and portfolios constructed with robust methods: affine equivariant robust estimators of mean and covariance matrix (S, MVE, MCD, SDE), pairwise covariance matrix

Received by the editors June5, 2012. Accepted by the editors May 19, 2013.
*Keywords*: Robust portfolios, Portfolio optimization, Robust estimation, Asset.
*JEL Classification*: C14, G11, G17.
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This paper is in final form and no version of it will be submitted for publication elsewhere.

(Spearman, Kendall, QC, OGK, I2D), as well as robust portfolio estimators (M, LAD). Theoretical portfolio properties and their investment effect will be compared. Apart from robust portfolios, portfolios created using other non-classical methods will be compared as well. Assessment of portfolio properties will be carried out on the basis of return rates of out-of-sample portfolios.

The author's own contribution is comparison, both on theoretical and practical grounds, of portfolios created using robust estimators. Moreover, a methodology of comparing alternative estimation methods, based on random portfolio selection, has been proposed, together with treating the out-of-sample risk and the out-of-sample return as investment effect of the portfolios.

The research to date, carried out by Lauprete (2001), Perret-Gentil and Victoria-Feser (2004), Mendes and Leal (2005), Welsch and Zhou (2007), DeMiguel and Nogales (2009), Grossi and Laurini (2011) indicate legitimacy of employing the robust methods in portfolio optimization. However, the results fail to explicitly indicate which robust methods allow to obtain the best investment results. The research was carried out on historical data or on simulated data, whereas an assessment of portfolio performance was often based not only on portfolio variance, but also on other parameters, such as Sharpe ratio or turnover.

The remainder of this paper is organised as follows. Section 2 contains an introduction to portfolio theory and shows possible problems with estimating mean-variance portfolios and describes robust estimators. Section 3 includes a theoretical comparison for properties of robust estimators. It also shows that creating robust portfolios with a rolling estimation window can lead to an increasing transaction cost. Section 4 describes the research hypothesis, the method of verifying the hypothesis, and a list of compared estimators. Section 5 features an empirical comparison between robust portfolios and classical portfolios, on the basis of the presented methodology. Section 6 concludes this paper

## 2. Robust estimation of portfolios

The classical approach to portfolio optimization, proposed by Markowitz (1952), consists in solving the following quadratic programming problem, which allows to define efficient portfolios:

$$\min_{w} \mathbf{w}'\mathbf{\Sigma}\mathbf{w}, \tag{2.1}$$

s.t.:

$$\begin{aligned} \mathbf{w}'\boldsymbol{\mu} &\geq \mu_0, \\ \mathbf{w}'\mathbf{1} &= 1, \end{aligned}$$

where $\boldsymbol{\mu} \in \mathcal{R}^p$ is the vector of expected returns for $p$ assets, $\mathbf{\Sigma}$ is covariance matrix, $\mathbf{w} \in \mathcal{R}^p$ is the vector of portfolio weights and $\mu_0$ is a target mean return. If short-selling is not allowed, an additional restriction $\mathbf{w} \geq 0$ is imposed.

It is well known that mean-variance portfolios (M-V) and market portfolios (as a special case of M-V portfolios) determined in the problem (2.1) have poor the out-of-sample performance due to a high estimation error, mainly in the scope of sample mean, whereas in this case the sample covariance matrix estimation error has lower impact (Black and Litterman,1992; Chopra and Ziemba, 1993; Jagannathan and Ma, 2003; DeMiguel and Nogales, 2009). The estimation error is significant enough that one should investigate only minimum-variance portfolios (Jagannathan and Ma, 2003), which are obtained by eliminating the $\mathbf{w}'\boldsymbol{\mu} \geq \mu_0$ restriction in the problem (2.1). The MVP portfolios are characterized by much lower estimation error (as they are based only on the sample covariance matrix) and their out-of-sample performance is much better than in the case of M-V portfolios (Jagannathan and Ma, 2003; DeMiguel and Nogales, 2009).

In the classical approach, the covariance matrix $\mathbf{\Sigma}$ in the problem (2.1) is replaced with matrix $\widehat{\mathbf{\Sigma}}_{MLE}$, which is a maximum likelihood estimator (MLE) under normal distribution. Stock returns are not normally distributed, which makes the MLE estimators inefficient in the case of even minor deviations from normal distribution, which is why the $\mathbf{\Sigma}$ matrix is often estimated using other estimation methods, which are less sensitive to both deviations from the assumed distribution (usually normal distribution), and to outliers.

In order to determine the portfolio structure, one needs to employ a consistent estimator of the variance-covariance matrix $\Sigma$. In the class of robust estimators there exist two-step approaches as *pairwise robust covariance estimators* and *affine equivariant robust estimators of mean and covariance matrix* and one-step approaches as *one-step robust portfolio estimators*.

To determine robust portfolios using two-step approach, it is necessary to replace the $\mathbf{\Sigma}$ matrix in the problem (2.1) with covariance matrix, for one-step approaches, the quadratic function in the problem (2.1) is replaced with function decreasing the impact of the outliers.

Pairwise robust covariance estimators. Pairwise covariance estimators are fast and easy to compute. One can distinguish three approaches to the determination of the covariance matrix (Alqallaf et al., 2002):

- Methods based on classical rank estimators - these methods apply classical rank estimators, such as Spearman's rho or Kendall's tau, described, for instance, in Abdullah (1990).
- Methods consisting in rejection of outliers for each random variable, followed by calculation of covariance for two variables - one example of such estimator can be quadrant correlation estimator (Maronna et al., 2006, section 6.9).
- Two-dimensional methods of rejecting outliers, such as Gnanadesikan-Kettenring Estimator (Gnanadesikan, Kettenring, 1972), 2D-Huber method (proposed by Khan et al., 2007), also known as Iterated bivariate Winsorization (Welsch and Zhou, 2007).

For the aforementioned methods, the obtained matrix is neither affine equivariant, nor positive definite. In order to obtain both these properties, an orthogonalization method, proposed by Maronna and Zamar (2002) is employed.

Affine equivariant robust estimators. Some of the most popular affine equivariant estimators are M-estimators proposed by Maronna (1976), S-estimators (Rousseeuw and Yohai, 1984), as well as Minimum Volume Ellipsoid (MVE) and Minimum Covariance Determinant (MCD) proposed by Rousseeuw (1984). This class contains also MM-estimators (Yohai, 1987), CM-estimators (Kent and Tayler, 1996), or Stahel-Donoho Estimator (SDE) defined independently by Stahel (1981) and Donoho (1982). Computation of affine equivariant estimators could be time-consuming, compared to classical estimators; for most of these methods, there is no exact algorithm, therefore the affine equivariance is often abandoned in favour of pairwise robust covariance estimators, which can be calculated much faster.

One-step robust portfolio estimators. In order to estimate the portfolio weight, the problem (2.1) can be rewritten equivalently:

$$\min_{w,m} \frac{1}{T} \sum_{t=1}^{T} \rho(\mathbf{w}'\mathbf{r}_t - m), \tag{2.2}$$

where $\rho(x) = x^2$, $T$ is a number of periods, $\mathbf{r}_t \in \mathcal{R}^p$ is the stock returns vector for $p$ assets within period $t$ and $m$ is the estimator of portfolio return which minimizes $\sum_{t=1}^{T} \rho(\mathbf{w}'\mathbf{r}_t - m)$. Similarly to linear regression or classical estimation of location and scale parameters, such a problem is susceptible to outliers. In order to reduce their impact, the quadratic functional form $(\cdot)^2$ is replaced by $\rho(\cdot)$, which allows to reduce the impact of atypical observations. In this group, the most popular estimators are LAD-portfolios and M-portfolios, which will be applied in the research.

LAD-portfolio - assuming that $\rho(\cdot) = |\cdot|$ in the problem (2.2), the resulting portfolio estimator is the least absolute deviation (LAD) estimator. In order to determine the LAD portfolio, it suffices to solve a linear programming problem proposed by Papahristodoulou and Dotzauer (2004). LAD-portfolios are a particular case of M-portfolios.

M-portfolio - if function $\rho$ is a convex symmetric function with unique minimum at 0, then the resulting estimator is an M-estimator of the portfolio. M-portfolios were proposed by Lauprete (2001), and investigated further by DeMiguel and Nogales (2009).

## 3. Properties of robust estimators - comparison

This section includes a theoretical comparison of the properties of robust estimators has been made. It also shows that creating robust portfolios with a rolling estimation window can lead to increasing transaction cost.

Contamination model. Two-step portfolios generated using affine equivariant estimators are more suitable when all rates of return for the companies analysed at the given day originate either from contaminating distribution, or from contaminated distribution (fully dependent contamination model, FDCM). Two-step portfolios generated using pairwise robust estimators are more suitable when contamination in stock returns distribution is independent for stock returns of each share (fully independent contamination model, FICM). The FICM and FDCM models are described in detail by Alqallaf et al. (2009). For one-step portfolios, according to the author's knowledge, no research comparing properties of these portfolios depending on the contamination model were carried out.

Sampling distribution. For affine equivariant estimators, it is assumed that the sampling distribution is a contaminated elliptical distribution. In the case of pairwise robust estimators, the sampling distribution does not have to be elliptical, due to the fact that separate estimation is carried out for each pair of random variables. To date, there has been no research concerning properties of portfolios generated using one-step method depending on sampling distribution.

Identification of outliers. Depending on the estimators, observation distances are identified in different ways, which changes the method of identifying outlying observations. In the case of pairwise robust methods, identification of outliers occurs either for each variable separately, or for each of the investigated pair of variables (depending on the method). For affine equivariant estimators, outliers are identified using Mahalanobis distance, or (in the case of an SDE estimator) on the basis of projection of observation. For portfolios generated using a one-step method, observations are identified using Euclidean distance from the data centre.

Algorithms. Some of the robust portfolios are additionally encumbered with an algorithm accuracy error, due to the fact that for some estimators the exact algorithm is unknown (e.g. SDE, MM), or precise computation of the estimator value is practically useful only for small samples (e.g. MCD). Precise calculation of the estimator value can be done for pairwise robust estimators and for affine equivariant M-estimators. In order to define robust portfolios, after calculating the covariance matrix and location parameter, a quadratic programming problem is solved (e.g. for minimum-variance portfolio). In the case of robust portfolios generated using one-step method, they are solved using non-linear programming problems (M-portfolios) or linear programming problems (LAD).

Transaction costs. Assuming that the investor creates rolling portfolios[1], it is important for the difference between weights determined at moment $t$ (on the basis of the last $n$ observations) and weights determined at moment $t+1$ (also on the basis of the last $n$ observations) to be as low as possible throughout the entire duration of the investment; therefore, it is important that a change in one observation does not significantly influence the weights on the portfolios. To simplify, one can investigate the difference between shares' weights determined at moment $t$ (on the basis of the last $n$ observations), and weights determined at moment $t+1$ (on the basis of the last $n+1$ observations), which results in an empirical influence function (also known as sensitivity curve), defined as follows:

$$EIF_{T_n}(\mathbf{x}) = (n+1)(T_{n+1}(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{x}_{n+1}) - T_n(\mathbf{x}_1,\ldots,\mathbf{x}_n)), \tag{3.1}$$

where $T_n$ is an estimator (i.e. estimator of portfolio weights) based on the $p$-dimensional random vector $(\mathbf{x}_1,...,\mathbf{x}_n)'$ from the sample of size $n$ (e.g. stock returns).

[1] Described in section 4.2.

The empirical influence function can be approximated by influence function ($IF$) for large sample (when $n$ tends to infinity, then $EIF$ tends to $IF$). It is also known (Perret-Gentil and Victoria-Feser, 2004) that for robust estimator an influence function is bounded. Unfortunately, decreasing sensitivity to outliers increases sensitivity to fewer observations (especially in a small sample). In practice, share of outliers is minor, thus for a rolling portfolio, most observations will cause greater changes in weights for robust portfolios than for classical ones[2], whereas only for a small number of observations (outlying ones), robust portfolios will be less sensitive than classic portfolios. Hence, as the breakdown point for the given estimator increases, its sensitivity for lesser observations grows, while its sensitivity to observations which are more distant from the bulk of data decreases.
Due to this problem, a high breakdown point estimators (especially pairwise estimators) cause much higher transaction cost than MLE estimators. This conclusion has been confirmed in empirical analysis in section 5.2.

## 4. Research methodology

4.1. **Compared estimators.** Classic portfolios generated using sample covariance matrix will be compared with the following robust methods: pairwise robust covariance estimators - Quadrant Correlation (QC), Orthogonalized Gnanadesikan-Kettenring (OGK), classical rank based methods - Kendall's tau (Kendall), Spearman's rho (Spearman) and Iterated bivariate Winsorization (I2D); affine equivariant robust estimators of mean and covariance matrix - S-estimators (computed using FastS), Rocke-type S estimators (Rocke), Minimum Volume Ellipsoid (MVE), Minimum Covariance Determinant (MCD) and Stahel-Donoho Estimator (SDE); one-step robust portfolios estimators - M-portfolios (M) and Least Absolute Deviation (LAD). For QC, OGK and I2D estimators, the following constants and functions are assumed:

- Quadrant correlation (QC) and Orthogonalized Gnanadesikan-Kettenring (OGK) - the scale parameter assumed for QC is median absolute deviation (MAD), whereas for OGK it is a "$\tau$-scale" function, analogically to Maronna and Zamar (2002). For both estimators, orthogonalization and estimator reweighting procedure is applied, with constant $\beta = 0.9$ - quantile of the $\chi^2_p$ distribution.
- Iterated bivariate Winsorization (I2D) - an algorithm analogical to (Welsch and Zhou, 2007) is applied, together with orthogonalization of the estimator.

Affine equivariant estimators is computed as per the following parameters:

- S-estimators with biweight function are calculated using FAST-S function (Todorov, Filzmoser, 2009) with breakdown point of 5% and 10%.
- Rocke type S-estimators proposed by Rocke (1996) with t-biweight function and asymptotic rejection point of 5% and 10%.
- Minimum Volume Ellipsoid (MVE) and Minimum Covariance Determinant (MCD) - the fraction of rejected observations is set at 5%, 10% and 20%. Additionally, an estimator reweighting procedure is applied, with constant $\beta = 0.975$ - quantile of the $\chi^2_p$ distribution.

For M-portfolios, Huber function is used as the $\rho$ function (analogically to DeMiguel and Nogales, 2009), the breakdown point is set at 5%, 10% and 20%.
Apart from robust methods, other methods of covariance matrix estimation are investigated, which, although non-robust to outliers, are robust to deviations of normal distribution and are characterized by lower estimation error. The first estimator (Student) is a maximum likelihood estimator for t-Student distributions, determined in accordance with algorithm proposed by Kent et al. (1994). The research assumes 5 degrees of freedom.
The next two estimators employ a shrinkage approach which is especially relevant for a small

[2] Classical portfolio has unbouded influence function

sample from high dimensional data sets: Leodoit-Wolf (L-W) estimator proposed by Ledoit and Wolf (2003) and Schäfer-Strimmer (S-S) estimator proposed by Schäfer and Strimmer (2005).

4.2. **Research procedure.** The research utilizes historical monthly rates of return for 71 companies from the FTSE 100 index, between February 2003 and November 2011 (106 months). The length of the analysed period (nearly 9 years) is a trade-off between number of assets and length of the period, because stretching time period the amount of assets decreases. Next, 200 portfolios consisting of 5, 10 or 15 companies are generated randomly[3]. The portfolios' structure is determined separately, using the methods previously described and only minimum variance portfolios are analysed, for the reasons advanced in section 2.

The portfolios are generated according to the following "rolling-horizon" procedure: portfolio weights are estimated at each month using the last 48 months and rebalanced every month. In the sample there are 106 monthly returns, so there are 58 rebalances. On the basis of the estimated weights, portfolio returns are calculated - 58 out-of-sample monthly returns. Then, the resulting rates of return are used to estimate the mean and standard deviation. As an estimator of standard deviation following estimators are considered: sample standard deviation (sd), mean absolute deviation (more robust than sd) and median absolute deviation (estimator with 50% breakdown point). Therefore, for each of the 200 $n$-asset portfolios (where $n$=5, 10, 15) generated using the given method, 200 values of estimated standard deviations and 200 values of estimated means are obtained, which are treated as the investment effect for each random portfolio.

Moreover, portfolio turnover was computed using the following formula (as in DeMiguel, Nogales, 2009):

$$T = \frac{1}{N-1} \sum_{t=1}^{N-1} \sum_{i=1}^{p} |w_i^{(t+1)} - w_i^{(t+)}|, \tag{4.1}$$

where $N$ is a number of changes in portfolio shares (in the research, it is 58), $p$ is a number of companies in the portfolio, $w_i^{(t+)}$ is the share of the $i$-th company at time $t$+1 before rebalancing, $w_i^{(t+1)}$ is a share of the $i$-th company in the portfolio at time $t+1$ (after rebalancing). The resulting value $T$ is an average part of the portfolio which undergoes reconstruction in each period.

4.3. **Research hypothesis.** This article proposes the following research hypothesis: appropriate application of robust estimators in construction of MVP portfolios allows to achieve better investment results (measured with standard deviation and expected rate of return) in comparison with portfolios defined using classic estimators.

As the basic goal of an investor who creates MVP portfolios is to minimize the portfolio risk. Hence it is verified whether the given method allows to significant reduction in portfolio risk, compared to the classical method. If the risk of the given method is not significantly different from the risk of the classical method, an additional verification criterion is applied to the resulting portfolio returns, and the portfolio turnover.

The risk of non-classical portfolios is compared to the risk of classical portfolios using Wilcoxon signed rank test[4], by comparing differences between estimated value of standard deviations of classical and non-classical portfolios. It was investigated whether the differences are actually higher than 0 (thus the risk of classical portfolios is higher) at significance level $\alpha$=5%/24 , where $\alpha$=5% is the significance level of the test procedure, whereas 24 is the number of non-classical methods compared to classical ones. This is due to obtain 5% significance level for the statistical procedure.

Similarly, it is examined whether the non-classical methods will allow to obtain higher average

[3]In the empirical research only small portfolios (less than 15 assets) are analysed. It is more practical to analyse small portfolios because average investor may construct only small portfolios due to wealth limitations.

[4]The Wilcoxon signed rank test for observation pairs is defined as sum of absolute values' ranks corresponding to positive differences.

returns than the classical portfolios, by comparing sample means. Yet, in this case, it is investigated whether the differences are significantly lower than 0 (which means that the average returns for classical portfolios are lower).

## 5. Empirical results

5.1. **Portfolio risk.** This section focuses on the analysis of standard deviations, calculated on the basis of the out-of-sample returns obtained by each of the 200 random portfolios, generated with the given method. As the measure of a risk, following estimators are applied: sample standard deviation, mean absolute deviation and median absolute deviation. Due to similar results obtained by this three measures, only results for sample standard deviation are discussed. Results for the last two measures are presented in appendix (Table 3 and Table 4 respectively). Table 1 presents a median of standard deviations and p-value of the Wilcoxon test for differences between standard deviations of portfolios calculated with the given method and of the classical portfolios. For all portfolios with nonnegative constraints on portfolio weights, there is no observable statistically significant improvement in the risk, whereas in the case of portfolios without nonnegative constraints (short-sale allowed), the risk decreases significantly only for 10- and 15-asset portfolios generated using L-W, Shrink, Student and I2D methods (15-stock portfolios only). For the I2D portfolios, the results confirm good properties demonstrated in research carried out by Welsch and Zhou (2007), albeit only if short sale is permitted; otherwise, these portfolios are characterized by a much higher risk than classical portfolios.

Table 1. Out-of-sample standard deviation

| | No short-sale | | | Short-sale allowed | | |
|---|---|---|---|---|---|---|
| Methods | 5 assets | 10 assets | 15 assets | 5 assets | 10 assets | 15 assets |
| I2D | 0.063 (1.00) | 0.053 (1.00) | 0.052 (1.00) | 0.064 (1.00) | 0.057 (0.51) | 0.056 (0.00)* |
| Kendall | 0.067 (1.00) | 0.058 (1.00) | 0.055 (1.00) | 0.067 (1.00) | 0.058 (0.90) | 0.055 (0.00)* |
| Classical | 0.055 (1.00) | 0.048 (1.00) | 0.045 (1.00) | 0.057 (1.00) | 0.054 (1.00) | 0.056 (1.00) |
| LAD | 0.057 (1.00) | 0.049 (1.00) | 0.046 (0.99) | 0.060 (1.00) | 0.055 (0.96) | 0.058 (0.99) |
| L-W | 0.056 (0.99) | 0.047 (0.08) | 0.045 (0.05) | 0.057 (0.02) | 0.049 (0.00)* | 0.048 (0.00)* |
| $MCD_{20\%}$ | 0.061 (1.00) | 0.050 (1.00) | 0.048 (1.00) | 0.065 (1.00) | 0.057 (0.98) | 0.061 (0.79) |
| $MCD_{10\%}$ | 0.060 (1.00) | 0.050 (1.00) | 0.048 (1.00) | 0.062 (1.00) | 0.056 (0.93) | 0.059 (0.44) |
| $MCD_{5\%}$ | 0.058 (1.00) | 0.049 (1.00) | 0.047 (1.00) | 0.060 (1.00) | 0.055 (0.62) | 0.060 (0.64) |
| $M_{20\%}$ | 0.056 (1.00) | 0.048 (1.00) | 0.046 (0.70) | 0.058 (0.92) | 0.055 (0.99) | 0.058 (1.00) |
| $M_{10\%}$ | 0.055 (1.00) | 0.048 (0.96) | 0.045 (0.23) | 0.058 (0.89) | 0.054 (0.94) | 0.056 (1.00) |
| $M_{5\%}$ | 0.055 (1.00) | 0.048 (0.83) | 0.045 (0.04) | 0.058 (0.88) | 0.054 (0.80) | 0.057 (0.99) |
| $MVE_{20\%}$ | 0.059 (1.00) | 0.050 (1.00) | 0.048 (1.00) | 0.061 (1.00) | 0.056 (0.98) | 0.059 (0.87) |
| $MVE_{10\%}$ | 0.057 (1.00) | 0.049 (1.00) | 0.047 (1.00) | 0.059 (1.00) | 0.055 (0.63) | 0.059 (0.63) |
| $MVE_{5\%}$ | 0.056 (1.00) | 0.049 (1.00) | 0.046 (1.00) | 0.058 (1.00) | 0.055 (0.93) | 0.058 (0.67) |
| OGK | 0.060 (1.00) | 0.050 (1.00) | 0.049 (1.00) | 0.061 (1.00) | 0.059 (0.88) | 0.059 (0.24) |
| QC | 0.062 (1.00) | 0.051 (1.00) | 0.048 (1.00) | 0.064 (1.00) | 0.058 (0.81) | 0.060 (0.32) |
| 1/N | 0.068 (1.00) | 0.063 (1.00) | 0.063 (1.00) | 0.068 (1.00) | 0.063 (1.00) | 0.063 (0.82) |
| SDE | 0.062 (1.00) | 0.051 (1.00) | 0.049 (1.00) | 0.065 (1.00) | 0.061 (1.00) | 0.071 (1.00) |
| $FastS_{10\%}$ | 0.071 (1.00) | 0.060 (1.00) | 0.054 (1.00) | 0.140 (1.00) | 0.255 (1.00) | 0.253 (1.00) |
| $FastS_{5\%}$ | 0.073 (1.00) | 0.059 (1.00) | 0.053 (1.00) | 0.168 (1.00) | 0.222 (1.00) | 0.232 (1.00) |
| Shrink | 0.060 (1.00) | 0.052 (1.00) | 0.051 (1.00) | 0.060 (1.00) | 0.054 (0.00)* | 0.052 (0.00)* |
| Spearman | 0.068 (1.00) | 0.057 (1.00) | 0.054 (1.00) | 0.068 (1.00) | 0.059 (0.87) | 0.060 (0.40) |
| $SRocke_{5\%}$ | 0.062 (1.00) | 0.053 (1.00) | 0.051 (1.00) | 0.066 (1.00) | 0.063 (1.00) | 0.077 (1.00) |
| $SRocke_{10\%}$ | 0.064 (1.00) | 0.053 (1.00) | 0.051 (1.00) | 0.069 (1.00) | 0.068 (1.00) | 0.079 (1.00) |
| Student | 0.055 (1.00) | 0.047 (0.33) | 0.045 (0.04) | 0.056 (0.63) | 0.053 (0.00)* | 0.055 (0.00)* |

*Notes*: The table presents a median of standard deviations for 200 randomly selected portfolios. In the brackets, there is the p-value for differences between standard deviations of the portfolios calculated with the given method and classical portfolios. An asterisk indicates p-values significant at significance level of $\alpha$=5%/24 , where $\alpha$=5% is the significance level of the test procedure.

Figure 1 - in appendix - presents a box plot graph for out-of-sample standard deviations. In the chart, it can be noted that portfolios with nonnegative constraints are characterized by

both a lower risk, and lower scattering of the risk compared to portfolios devoid of these constraints. Comparison of portfolios generated without nonnegative constraints shows that for most portfolios, increasing the number of assets to 15 caused both increase in the risk and greater scattering of the portfolio risk. These observations confirm results of the research to date, which demonstrate that imposition of nonnegative constraints allows to restrict the risk of the investigated portfolios and to diversify the portfolio risk while increasing the number of assets. Nonetheless, for these portfolios, none of the methods examined allows to decrease the risk of the investigated portfolios. Therefore, if the primary goal of the investor is to minimize the portfolio risk, classical portfolios are the right choice.

Figure 2 - in appendix - presents a box plot graph of differences between standard deviations of portfolios calculated with the given method and classical portfolios. The portfolios presented are 15-stock portfolios with short-selling constraints. Among the investigated estimators, only portfolios generated with L-W and Student portfolios, as well as M-portfolios and LAD-portfolios achieved a level of risk similar to classical portfolios, whereas the risk of the remaining portfolios was significantly higher.

Analysing performance of robust portfolios, one can observe that decreasing the breakdown point of robust portfolios allows to decrease the risk compared with the risk of classical portfolios. It can also be noticed that despite theoretically better properties of the MCD estimator relative to MVE (e.g. better effectiveness), portfolios created with these methods achieve very similar results. S-portfolios calculated using the FAST-S algorithm are characterized by a much higher risk than other methods, therefore in the case of application in the portfolio theory, this algorithm is unsuitable. The research demonstrates that portfolios generated with pairwise robust covariance estimators for portfolios with nonnegative constraints achieve higher risk than other robust portfolios, yet after removing these constraints, the risk of these portfolios is much lower. This means that in the case of analysing scattering of returns, these estimators should be better, whereas in the instance of creating portfolios with nonnegative constraints, portfolio estimators created using one-step approach (LAD-portfolios and M-portfolios) are more suitable.

To sum up, it can be seen that minimum variance portfolios and application of nonnegative constraints allow to achieve lower risk than when these constraints are absent. It could be surprising that solution of a more constrained minimization problem is less than a solution of the relaxed problem, but this is only true in a sample while presented results are based on the out-of-sample returns. It is also in accordance with results obtained by Jagannathan and Ma (2003) who show that imposing nonnegative constraint reduce the sampling error.

Moreover, it can be noticed that none of the analysed methods allows to decrease the portfolio risk compared to the risk of classical portfolios. For portfolios with nonnegative constraints, only L-W and Student methods, as well as one-step robust portfolios, had risk level similar to classical portfolios, whereas the remaining methods caused significant increase in the portfolio risk. Nonetheless, for affine equivariant methods MCD and MVE, the risk increase is slightly greater (Figure 2).

5.2. **Portfolio returns and turnover.** This section focuses on analysis of an average monthly rate of returns and portfolio turnovers, calculated on the basis of out-of-sample returns obtained by each of the 200 random portfolios, generated with the given method. The presented results concern only portfolios with nonnegative constraints. Table 2 presents a median of average returns for each portfolio, as well as p-value of the Wilcoxon test for differences between the portfolios returns calculated with the given method and classical portfolios. The second part of the table presents portfolio turnover. The presented p-value for Wilcoxon test for observation pairs allows to see whether the average monthly returns for the investigated portfolios are significantly higher than the average returns for classical portfolios. At the significance level of $\alpha$=5%/24, only L-W and 5-stock M-portfolios fail to show a significant difference, other portfolios have significantly higher returns. Other methods having achieved a risk level similar

to classical portfolios (LAD-portfolios, Student, 10- and 15-stock M-portfolios) allow to obtain higher returns. MCD and MVE portfolios having been characterized by slightly higher risk than classical portfolios, allow to achieve substantially higher returns. Comparison of differences between average monthly returns for the examined portfolios and classical portfolios is illustrated as Figure 3 - in appendix.
Comparing portfolio turnover values, one can observe that for most portfolios, increasing the number of assets from 5 to 15 has caused an almost twofold increase in these values. Classical portfolios and portfolios created using shrinkage methods (L-W, S-S), rank methods, Student and M-portfolios have the lowest turnover. Portfolios generated using most of the affine equivariant robust estimators are characterized by slightly higher costs, whereas the remaining pairwise robust covariance estimators and LAD-portfolios have the highest turnover. For some methods, turnover can have a significant impact on the returns, for instance, if the commission rate is 0,2%, the returns for LAD portfolios will drop by about 0,11% on average, which in turn will allow these portfolios to achieve returns similar to classical portfolios.

Table 2. Characteristics of portfolios with nonnegative constraints

| Methods | Returns | | | Turnover | | |
|---|---|---|---|---|---|---|
| | 5 assets | 10 assets | 15 assets | 5 assets | 10 assets | 15 assets |
| I2D | 0,38% (0,00) | 0,35% (0,00) | 0,28% (0,00) | 0,192 | 0,268 | 0,309 |
| Kendall | 0,66% (0,00) | 0,60% (0,00) | 0,49% (0,00) | 0,061 | 0,086 | 0,106 |
| Classical | 0,19% (1,00) | 0,14% (1,00) | 0,16% (1,00) | 0,079 | 0,113 | 0,135 |
| LAD | 0,24% (0,00) | 0,22% (0,00) | 0,24% (0,00) | 0,243 | 0,41 | 0,531 |
| L-W | 0,19% (1,00) | 0,02% (1,00) | 0,02% (1,00) | 0,066 | 0,084 | 0,095 |
| $MCD_{20\%}$ | 0,46% (0,00) | 0,39% (0,00) | 0,35% (0,00) | 0,125 | 0,195 | 0,233 |
| $MCD_{10\%}$ | 0,41% (0,00) | 0,24% (0,00) | 0,26% (0,00) | 0,115 | 0,166 | 0,198 |
| $MCD_{5\%}$ | 0,35% (0,00) | 0,24% (0,00) | 0,19% (0,00) | 0,107 | 0,144 | 0,17 |
| $M_{20\%}$ | 0,23% (0,00) | 0,19% (0,00) | 0,25% (0,00) | 0,088 | 0,131 | 0,159 |
| $M_{10\%}\%$ | 0,22% (0,01) | 0,16% (0,00) | 0,20% (0,00) | 0,084 | 0,123 | 0,15 |
| $M_{5\%}$ | 0,20% (0,05) | 0,13% (0,00) | 0,20% (0,00) | 0,082 | 0,12 | 0,145 |
| $MVE_{20\%}$ | 0,40% (0,00) | 0,29% (0,00) | 0,32% (0,00) | 0,194 | 0,324 | 0,399 |
| $MVE_{10\%}$ | 0,30% (0,00) | 0,23% (0,00) | 0,24% (0,00) | 0,16 | 0,245 | 0,307 |
| $MVE_{5\%}$ | 0,23% (0,00) | 0,16% (0,00) | 0,19% (0,00) | 0,143 | 0,196 | 0,231 |
| OGK | 0,41% (0,00) | 0,40% (0,00) | 0,46% (0,00) | 0,165 | 0,259 | 0,312 |
| QC | 0,47% (0,00) | 0,46% (0,00) | 0,48% (0,00) | 0,218 | 0,32 | 0,376 |
| 1/N | 0,64% (0,00) | 0,66% (0,00) | 0,68% (0,00) | 0,056 | 0,058 | 0,06 |
| SDE | 0,48% (0,00) | 0,47% (0,00) | 0,48% (0,00) | 0,174 | 0,288 | 0,333 |
| $SFast_{10\%}$ | 0,57% (0,00) | 0,62% (0,00) | 0,58% (0,00) | 0,448 | 0,669 | 0,632 |
| $SFast_{5\%}$ | 0,63% (0,00) | 0,61% (0,00) | 0,57% (0,00) | 0,588 | 0,664 | 0,598 |
| S-S | 0,40% (0,00) | 0,37% (0,00) | 0,33% (0,00) | 0,068 | 0,09 | 0,11 |
| Spearman | 0,64% (0,00) | 0,52% (0,00) | 0,47% (0,00) | 0,076 | 0,117 | 0,142 |
| $SRocke_{5\%}$ | 0,49% (0,00) | 0,40% (0,00) | 0,41% (0,00) | 0,171 | 0,403 | 0,649 |
| $SRocke_{10\%}$ | 0,46% (0,00) | 0,43% (0,00) | 0,40% (0,00) | 0,207 | 0,553 | 0,715 |
| Student | 0,31% (0,00) | 0,27% (0,00) | 0,27% (0,00) | 0,079 | 0,113 | 0,135 |

*Notes*: The table presents a median of portfolios' characteristics calculated on the basis of out-of-sample rates of return obtained by each of the 200 random portfolios generated with the given method: average monthly rate of returns and portfolio turnover. The values are presented separately for 5-, 10- and 15-stock portfolios. In the brackets, there is the p-value for differences between standard deviations of the portfolios calculated with the given method and classical portfolios.

Comparing the turnover for robust estimators, one can observe that as the breakdown point decreases, the turnover drops as well. This observation results from properties of robust estimators, which, due to lower susceptibility to outliers, are more susceptible to less distant observations, which occur much more often.
In the group of portfolios which achieves a risk level similar to classical portfolios, after considering turnover and returns, it can be noticed that only M-portfolios and Student portfolios allow

to achieve a higher returns. MVE and MCD portfolios, which are slightly more risky than classical portfolios, allow to obtain much higher returns, particularly in the case of smaller, 5-asset portfolios.

## 6. Conclusion

This paper describes and compares three groups of robust estimators: pairwise robust covariance estimators, affine equivariant robust estimators of mean and covariance matrix and one-step robust portfolios estimators. The first part of this paper compares theoretical differences between the aforementioned groups of estimators and indicates the most important differences. The next one, compares a methodology of assessing investment results for global minimum variance portfolios created using non-classical methods are proposed. The analysis is based on generating portfolios with randomly selected assets, followed by comparing their characteristics calculated on the basis of out-of-sample returns.
It can be seen that for MVP portfolios created using robust methods, there is no significant risk improvement, and that even for most robust methods (except for M-portfolios and LAD-portfolios), there is an observable significant risk increase compared to the risk of classical portfolios. For M-portfolios, a significant improvement in returns with slight increase in the turnover can be observed, whereas the LAD-portfolios are characterized by such a high turnover that in practice they do not allow to obtain higher returns than classical portfolios, due to transaction costs. Despite having a slightly higher risk, portfolios created using MVE and MCD estimators allow to achieve much higher returns, particularly for small portfolios. It can be noticed that increasing the breakdown point of robust portfolios causes decrease in both the risk and the turnover.
Moreover, portfolios generated using shrinkage estimators, as well as Student estimator, are compared. The former fails to demonstrate a significant risk improvement, whereas the Student portfolios are characterized by properties similar to M-portfolios.
Empirical research also confirms previous research findings: MVP portfolios and application of nonnegative constraints allow to achieve lower risk than when these constraints are absent.
There is a set of reasons for which the previously reported results differ from the existing ones: presented methodology uses random sets of assets instead of one set, empirical comparison includes real data set and finally, portfolio performance are analysed out-of-sample. What is more, assuming occurrence of outliers in sample follows occurrence of outliers out-of-sample. Thus, true efficient frontier is not based on real covariance matrix from uncontaminated distribution $\mathbf{F}$, but it is based on covariance matrix from mixture model, where most of (but not all) observations come from uncontaminaited distribution $\mathbf{F}$.

**Acknowledgement.** *This project was financed from the funds of the National Science Centre (Poland).*

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7.

## Appendix - Boxplots

Table 3. Out-of-sample mean absolute deviation

| | No short-sale | | | Short-sale allowed | | |
|---|---|---|---|---|---|---|
| Methods | 5 assets | 10 assets | 15 assets | 5 assets | 10 assets | 15 assets |
| I2D | 0.046 (1.00) | 0.040 (1.00) | 0.040 (1.00) | 0.047 (1.00) | 0.042 (1.00) | 0.042 (0.31) |
| Kendall | 0.051 (1.00) | 0.044 (1.00) | 0.042 (1.00) | 0.051 (1.00) | 0.044 (1.00) | 0.043 (0.76) |
| Classical | 0.041 (1.00) | 0.037 (1.00) | 0.035 (1.00) | 0.042 (1.00) | 0.040 (1.00) | 0.041 (1.00) |
| LAD | 0.042 (1.00) | 0.038 (1.00) | 0.036 (1.00) | 0.044 (1.00) | 0.041 (1.00) | 0.043 (1.00) |
| L-W | 0.042 (1.00) | 0.036 (0.00)* | 0.035 (0.00)* | 0.042 (0.01) | 0.037 (0.00)* | 0.035 (0.00)* |
| $MCD_{20\%}$ | 0.045 (1.00) | 0.039 (1.00) | 0.037 (1.00) | 0.047 (1.00) | 0.042 (1.00) | 0.045 (1.00) |
| $MCD_{10\%}$ | 0.044 (1.00) | 0.039 (1.00) | 0.037 (1.00) | 0.045 (1.00) | 0.042 (1.00) | 0.043 (1.00) |
| $MCD_{5\%}$ | 0.043 (1.00) | 0.038 (1.00) | 0.037 (1.00) | 0.044 (1.00) | 0.041 (1.00) | 0.043 (1.00) |
| $M_{20\%}$ | 0.042 (1.00) | 0.037 (1.00) | 0.035 (0.70) | 0.043 (0.98) | 0.041 (1.00) | 0.041 (1.00) |
| $M_{10\%}$ | 0.042 (1.00) | 0.037 (0.96) | 0.035 (0.28) | 0.043 (0.95) | 0.040 (1.00) | 0.041 (1.00) |
| $M_{5\%}$ | 0.042 (1.00) | 0.037 (0.84) | 0.035 (0.08) | 0.043 (0.97) | 0.040 (0.99) | 0.041 (1.00) |
| $MVE_{20\%}$ | 0.044 (1.00) | 0.038 (1.00) | 0.037 (1.00) | 0.045 (1.00) | 0.042 (1.00) | 0.044 (1.00) |
| $MVE_{10\%}$ | 0.043 (1.00) | 0.038 (1.00) | 0.036 (1.00) | 0.045 (1.00) | 0.041 (1.00) | 0.043 (1.00) |
| $MVE_{5\%}$ | 0.042 (1.00) | 0.037 (1.00) | 0.036 (1.00) | 0.043 (1.00) | 0.041 (1.00) | 0.042 (1.00) |
| OGK | 0.045 (1.00) | 0.039 (1.00) | 0.038 (1.00) | 0.046 (1.00) | 0.043 (1.00) | 0.043 (1.00) |
| QC | 0.046 (1.00) | 0.039 (1.00) | 0.037 (1.00) | 0.048 (1.00) | 0.043 (1.00) | 0.044 (1.00) |
| 1/N | 0.051 (1.00) | 0.047 (1.00) | 0.047 (1.00) | 0.051 (1.00) | 0.047 (1.00) | 0.047 (1.00) |
| SDE | 0.047 (1.00) | 0.040 (1.00) | 0.038 (1.00) | 0.048 (1.00) | 0.045 (1.00) | 0.052 (1.00) |
| $FastS_{10\%}$ | 0.052 (1.00) | 0.045 (1.00) | 0.040 (1.00) | 0.095 (1.00) | 0.150 (1.00) | 0.158 (1.00) |
| $FastS_{5\%}$ | 0.053 (1.00) | 0.044 (1.00) | 0.040 (1.00) | 0.111 (1.00) | 0.138 (1.00) | 0.148 (1.00) |
| Shrink | 0.045 (1.00) | 0.040 (1.00) | 0.039 (1.00) | 0.045 (1.00) | 0.041 (0.04) | 0.039 (0.00)* |
| Spearman | 0.050 (1.00) | 0.044 (1.00) | 0.042 (1.00) | 0.050 (1.00) | 0.046 (1.00) | 0.046 (1.00) |
| $SRocke_{5\%}$ | 0.046 (1.00) | 0.040 (1.00) | 0.039 (1.00) | 0.048 (1.00) | 0.045 (1.00) | 0.055 (1.00) |
| $SRocke_{10\%}$ | 0.047 (1.00) | 0.041 (1.00) | 0.039 (1.00) | 0.050 (1.00) | 0.051 (1.00) | 0.056 (1.00) |
| Student | 0.042 (1.00) | 0.037 (1.00) | 0.036 (0.97) | 0.043 (0.98) | 0.040 (0.00)* | 0.040 (0.00)* |

*Notes*: The table presents a median of mean absolute deviations for 200 randomly selected portfolios. In the brackets, there is the p-value for differences between mean absolute deviations of the portfolios calculated with the given method and classical portfolios. An asterisk indicates p-values significant at significance level of $\alpha$=5%/24 , where $\alpha$=5% is the significance level of the test procedure.

TABLE 4. Out-of-sample median absolute deviation

| | No short-sale | | | Short-sale allowed | | |
|---|---|---|---|---|---|---|
| Methods | 5 assets | 10 assets | 15 assets | 5 assets | 10 assets | 15 assets |
| I2D | 0.051 (1.00) | 0.047 (1.00) | 0.046 (1.00) | 0.052 (1.00) | 0.047 (1.00) | 0.047 (1.00) |
| Kendall | 0.055 (1.00) | 0.052 (1.00) | 0.050 (1.00) | 0.055 (1.00) | 0.052 (1.00) | 0.050 (1.00) |
| Classical | 0.048 (1.00) | 0.044 (1.00) | 0.042 (1.00) | 0.048 (1.00) | 0.045 (1.00) | 0.044 (1.00) |
| LAD | 0.049 (1.00) | 0.044 (0.97) | 0.043 (0.96) | 0.049 (0.78) | 0.046 (1.00) | 0.046 (1.00) |
| L-W | 0.048 (0.13) | 0.043 (0.00)* | 0.042 (0.00)* | 0.048 (0.02) | 0.042 (0.00)* | 0.040 (0.00)* |
| $MCD_{20\%}$ | 0.049 (1.00) | 0.046 (1.00) | 0.045 (1.00) | 0.051 (1.00) | 0.048 (1.00) | 0.049 (1.00) |
| $MCD_{10\%}$ | 0.049 (1.00) | 0.046 (1.00) | 0.044 (1.00) | 0.050 (1.00) | 0.048 (1.00) | 0.047 (1.00) |
| $MCD_{5\%}$ | 0.049 (0.97) | 0.046 (1.00) | 0.044 (1.00) | 0.050 (0.99) | 0.047 (1.00) | 0.047 (1.00) |
| $M_{20\%}$ | 0.048 (0.14) | 0.043 (0.32) | 0.042 (0.20) | 0.048 (0.02) | 0.045 (0.98) | 0.045 (0.87) |
| $M_{10\%}$ | 0.047 (0.08) | 0.043 (0.24) | 0.042 (0.33) | 0.048 (0.01) | 0.045 (0.95) | 0.045 (0.70) |
| $M_{5\%}$ | 0.047 (0.23) | 0.044 (0.32) | 0.042 (0.23) | 0.048 (0.03) | 0.045 (0.99) | 0.044 (0.33) |
| $MVE_{20\%}$ | 0.050 (1.00) | 0.046 (1.00) | 0.044 (1.00) | 0.050 (1.00) | 0.048 (1.00) | 0.049 (1.00) |
| $MVE_{10\%}$ | 0.048 (0.88) | 0.045 (1.00) | 0.043 (1.00) | 0.049 (1.00) | 0.047 (1.00) | 0.047 (1.00) |
| $MVE_{5\%}$ | 0.048 (0.70) | 0.045 (0.99) | 0.043 (0.78) | 0.049 (0.93) | 0.046 (1.00) | 0.046 (1.00) |
| OGK | 0.051 (1.00) | 0.047 (1.00) | 0.044 (1.00) | 0.051 (1.00) | 0.048 (1.00) | 0.048 (1.00) |
| QC | 0.051 (1.00) | 0.047 (1.00) | 0.046 (1.00) | 0.051 (1.00) | 0.048 (1.00) | 0.049 (1.00) |
| 1/N | 0.056 (1.00) | 0.052 (1.00) | 0.051 (1.00) | 0.056 (1.00) | 0.052 (1.00) | 0.051 (1.00) |
| SDE | 0.052 (1.00) | 0.046 (1.00) | 0.045 (1.00) | 0.051 (1.00) | 0.050 (1.00) | 0.057 (1.00) |
| $FastS_{10\%}$ | 0.058 (1.00) | 0.051 (1.00) | 0.048 (1.00) | 0.092 (1.00) | 0.130 (1.00) | 0.139 (1.00) |
| $FastS_{5\%}$ | 0.057 (1.00) | 0.050 (1.00) | 0.047 (1.00) | 0.099 (1.00) | 0.120 (1.00) | 0.126 (1.00) |
| Shrink | 0.051 (1.00) | 0.047 (1.00) | 0.046 (1.00) | 0.051 (1.00) | 0.048 (1.00) | 0.046 (0.99) |
| Spearman | 0.056 (1.00) | 0.052 (1.00) | 0.051 (1.00) | 0.056 (1.00) | 0.054 (1.00) | 0.054 (1.00) |
| $SRocke_{5\%}$ | 0.052 (1.00) | 0.047 (1.00) | 0.046 (1.00) | 0.052 (1.00) | 0.050 (1.00) | 0.057 (1.00) |
| $SRocke_{10\%}$ | 0.052 (1.00) | 0.048 (1.00) | 0.046 (1.00) | 0.053 (1.00) | 0.055 (1.00) | 0.059 (1.00) |
| Student | 0.049 (0.80) | 0.044 (0.96) | 0.043 (0.91) | 0.049 (0.83) | 0.045 (1.00) | 0.044 (0.69) |

*Notes*: The table presents a median of median absolute deviations for 200 randomly selected portfolios. In the brackets, there is the p-value for differences between median absolute deviations of the portfolios calculated with the given method and classical portfolios. An asterisk indicates p-values significant at significance level of $\alpha$=5%/24 , where $\alpha$=5% is the significance level of the test procedure.

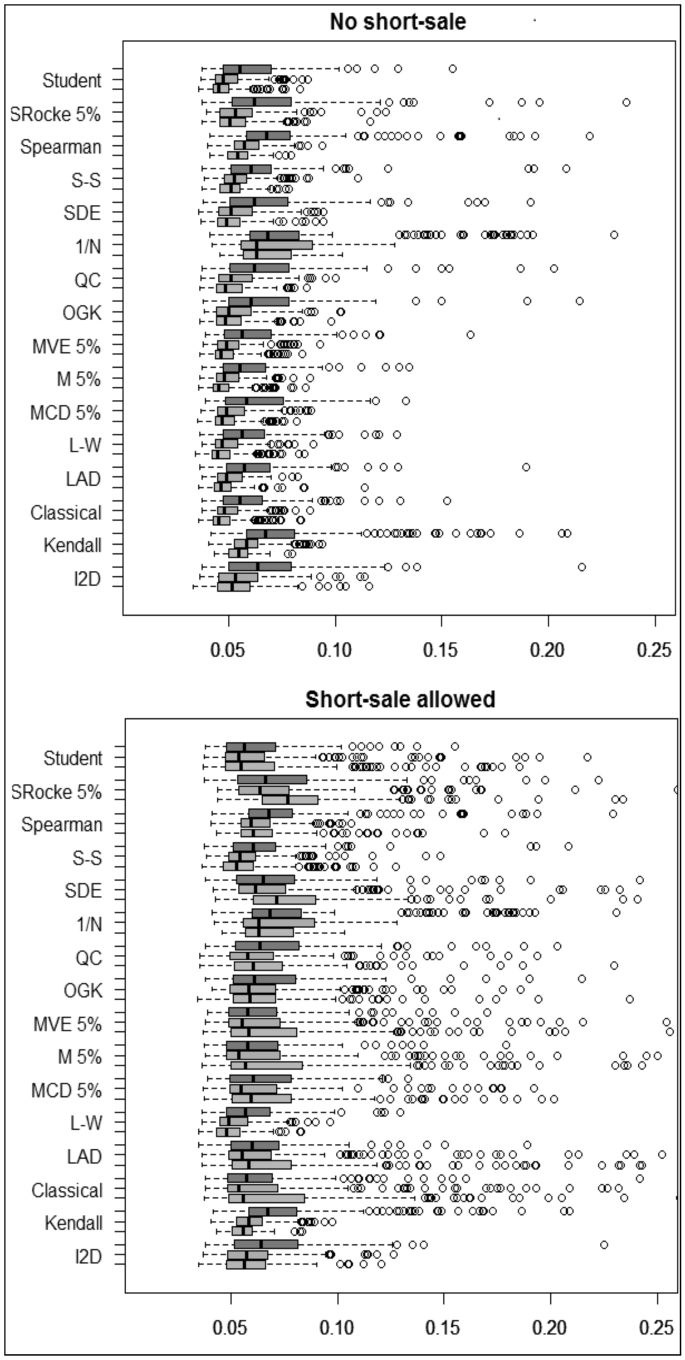


FIGURE 1. **Boxplots of standard deviations.** Each boxplot illustrates standard out-of-sample deviations for 200 randomly generated portfolios. The upper panel presents portfolios with nonnegative constraints, the lower portfolios without nonnegative constraints. Upper boxes present 5-asset portfolios, middle boxes - 10-asset portfolio and lower boxes - 15-asset portfolio.

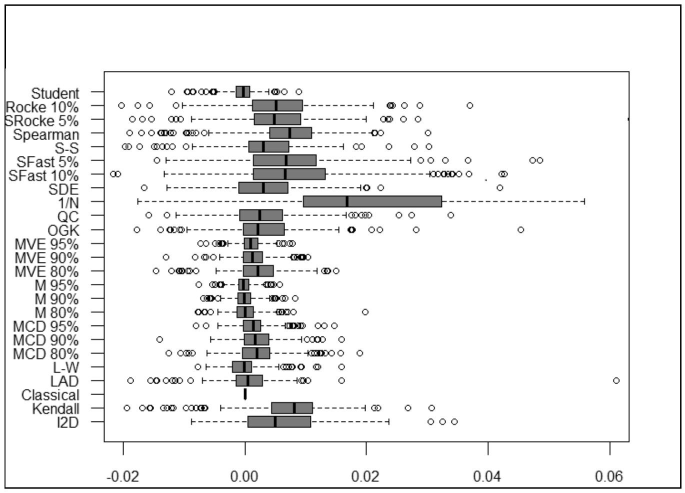


FIGURE 2. **Boxplots of differences between standard deviations of classical portfolios and non-classical portfolios with nonnegative constraints**. This chart presents differences between standard deviations of classical portfolios and portfolios calculated using the given method. For each of the two hundred 15-stock portfolios, the standard deviation was computed on the basis of out-of-sample returns. Positive values indicate higher risk of classical portfolios.

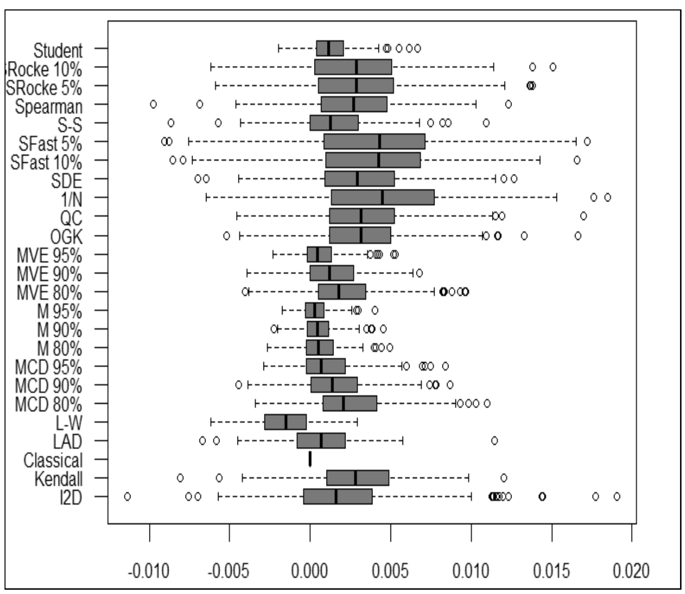


FIGURE 3. **Boxplot of differences between average monthly returns of classical and non-classical portfolios with nonnegative constraints**. This chart presents differences between average monthly returns of classical portfolios and portfolios calculated out of sample. For each of the two hundred 15-stock portfolios, average monthly returns were computed on the basis of out-of-sample returns. Positive values indicate higher returns of non-classical portfolios.